The Birth of Mathematics: Ancient Times to 1300

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PREFACE

Mathematics is a human endeavor. Behind its numbers, equations, formulas, and theorems are the stories of the people who expanded the frontiers of humanity's mathematical knowledge. Some were child prodigies while others developed their aptitudes for mathematics later in life. They were rich and poor, male and female, well educated and self-taught. They worked as professors, clerks, farmers, engineers, astronomers, nurses, and philosophers. The diversity of their backgrounds testifies that mathematical talent is independent of nationality, ethnicity, religion, class, gender, or disability.

Pioneers in Mathematics is a five-volume set that profiles the lives of 50 individuals, each of whom played a role in the development and the advancement of mathematics. The overall profiles do not represent the 50 most notable mathematicians; rather, they are a collection of individuals whose life stories and significant contributions to mathematics will interest and inform middle school and high school students. Collectively, they represent the diverse talents of the millions of people, both anonymous and well known, who developed new techniques, discovered innovative ideas, and extended known mathematical theories while facing challenges and overcoming obstacles.

Each book in the set presents the lives and accomplishments of 10 mathematicians who lived during an historical period. *The Birth of Mathematics* profiles individuals from ancient Greece, India, Arabia, and medieval Italy who lived from 700 B.C.E. to 1300 C.E. *The Age of Genius* features mathematicians from Iran, France, England, Germany, Switzerland, and America who lived between geometrical techniques to estimate perimeters, areas, and volumes, to determine tangent lines, and to trisect angles. In the fourth century C.E., Hypatia of Alexandria, the earliest-known woman to write and teach about higher mathematics, wrote commentaries that enhanced and preserved the works of earlier Greek scholars.

Generations of mathematicians in India also developed advanced ideas and techniques in various branches of mathematics. Two of the foremost Hindu scholars of this period were Āryabhata and Brahmagupta. In the sixth century, Āryabhata introduced an alphabetical system of notation to represent large numbers and developed techniques for estimating distances, determining areas, and calculating volumes. In the seventh century, Brahmagupta developed rules for performing arithmetic with negative numbers and introduced iterative algorithms to find sines of angles and square roots.

During the next six centuries, Arabic mathematicians further extended the discoveries of Greek and Indian scholars. The ninth-century mathematician Muhammad al-Khwārizmī demonstrated how to solve second-degree equations in the earliest-known text on algebra. In the 11th century, Omar Khayyám developed geometrical techniques for solving algebraic equations and expanded on Euclid's theory of ratios.

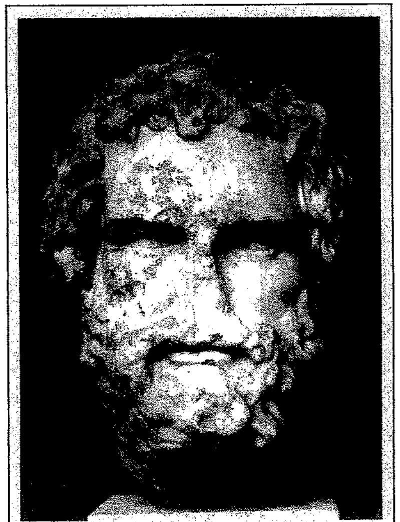
In 13th-century Italy, Leonardo Fibonacci wrote about the base-10 number system and the efficient computational algorithms that Hindu and Arabic scholars had developed. His book was one of several similar works on arithmetic and computation that caused western Europeans to renew their interest in Greek mathematics and convinced them to adopt the Hindu-Arabic numbering system.

These 10 mathematicians made additional significant contributions to the progress of mathematics and science. Thousands more of their colleagues and countrymen made important mathematical discoveries that advanced the world's knowledge. The stories of their achievements provide a glimpse into the lives and the minds of some of the pioneers who discovered mathematics.

1

Thales of Miletus

(ca. 625–ca. 547 B.C.E.)



Thales of Miletus proved the earliest theorems in geometry. *(The Granger Collection)*

Earliest Proofs of Geometrical Theorems

Thales (pronounced THAY-leez) of Miletus established the study of natural philosophy in a world dominated by Greek mythology. His proofs of five theorems in geometry introduced the concept of logical theory into mathematics. As an astronomer, he predicted a solar eclipse and improved the existing techniques for navigating by the stars. Thales became known throughout the ancient Greek world for his ingenious solutions to practical problems involving pyramids, donkeys, rivers, and ships.

Early Years

Conflicting historical records place the date of Thales' birth between 641 and 625 B.C.E., although the later date is generally accepted as more accurate. He was born in Miletus, a small town located 200 miles east of Athens across the Aegean Sea in the Greek province of Ionia in present-day Turkey. Miletus was a seaport on the trade routes that linked the Mediterranean world with India and other countries of the Near East. As Thales traveled outside his local community, he became known as Thales of Miletus.

Little is known about Thales' family or his early life. Cleobuline and Examyes, his mother and father, respectively, were a distinguished family, but their careers and achievements are not known. As a young man, Thales traveled to Egypt and Babylonia (modern-day Iraq) to pursue his interests in astronomy, mathematics, and science. He learned how Egyptians used practical geometrical techniques to measure distances and to calculate areas of plots of farmland. He studied Babylonian astronomy and its use of a base-60 number system.

Natural Philosopher

Around 590 B.C.E., Thales returned to Miletus and established a school known as the Ionian School of Philosophy, where he taught science, astronomy, mathematics, and philosophy. In his philosophy classes, he shared with his students his ideas about the meaning of life and the love of wisdom. He stressed the importance of asking questions, especially the question "Why?" In all areas of study, he emphasized that the workings of the world could be explained in terms of logical, underlying theories.

At the time, Greeks believed that their lives were determined by the actions of many gods. According to their mythology, the god Demeter made crops and animals grow; the god Dionysus made wine taste sweet or bitter; the goddess Aphrodite made people fall in love; the god Ares decided who won wars. Thales did not accept

stories about gods as explanations for why events occurred. He was convinced that there had to be natural reasons to explain why the world behaved as it did.

Like the people of his day, Thales believed the Earth was a large disk floating on an underground ocean of water. According to a Greek myth, the god Poseidon, who lived in this underground ocean, would shake the Earth, causing an earthquake when he was angry. Searching for a more logical and natural explanation, Thales reasoned that if waves in the sea could rock boats back and forth, then waves in this underground ocean could push against the ground from below, causing it to shake. He taught this theory to the students at his school and encouraged them to seek similar explanations for other physical occurrences.

Although Thales' theory about the cause of earthquakes was not correct, his search for natural rather than supernatural or mystical reasons to explain such events was a radically new approach to understanding the world. His insistence on natural explanations and unifying theories that linked a cause with its effect became known as natural philosophy. Aristotle, in his book *Metaphysics*, credited Thales as being the founder of Ionian natural philosophy. By searching for the laws of nature that explained physical phenomena, Thales paved the way for the development of science.

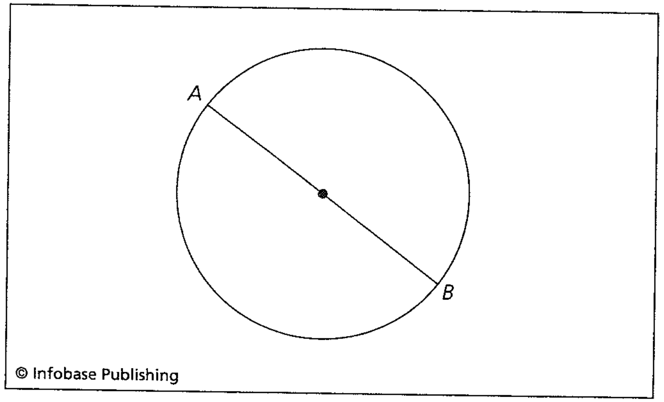
First Proofs of Theorems in Mathematics

In his school, Thales taught that mathematical ideas were logically connected to each other rather than being a collection of unrelated rules. He also believed that mathematical results were true not only because they agreed with our experiences of the world around us but also for deeper reasons. Thales searched for a set of basic principles and the logic that would allow him to develop all mathematical properties and rules from them. He called these basic principles axioms and postulates. A property that could be obtained from them by a logical argument was called a theorem, and the logical reasoning was called a proof.

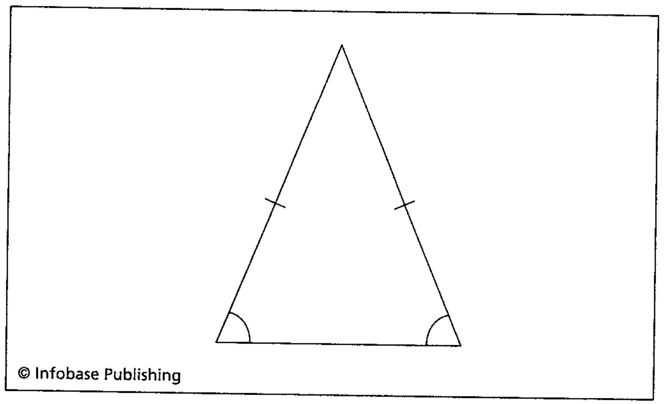
Thales proved five theorems about geometrical properties of circles and triangles. These results were known to be true, but no one had explained *why* they were true. Thales showed how they followed logically from the basic axioms of geometry.

The following are the five theorems Thales proved:

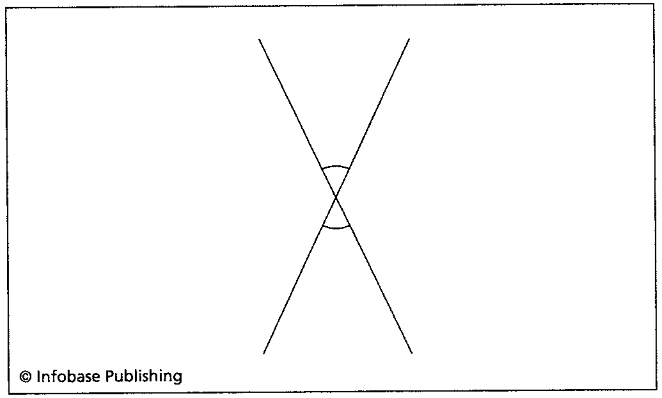
1. Any line drawn through the center of a circle will divide that circle into two equal areas. In other words, any diameter bisects the circle.



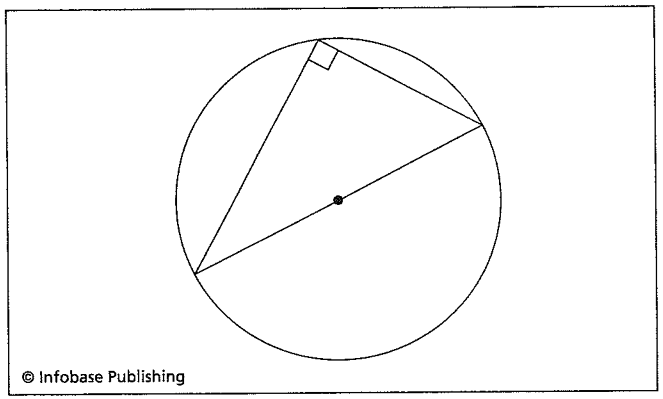
2. If a triangle has two sides that are equal in length, then the two angles opposite those sides must also be equal in measure. That is to say, the base angles of an isosceles triangle are equal.



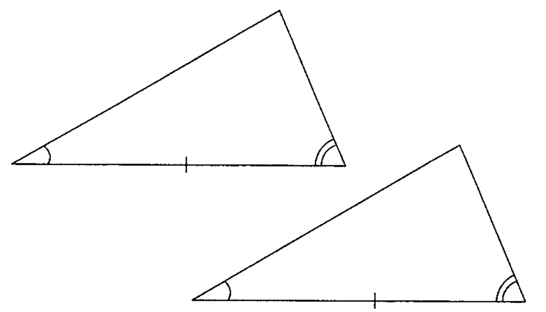
3. If two lines intersect, each pair of angles that open in opposite directions must be equal to each other. More efficiently stated, vertical angles formed by intersecting lines are equal.



4. If the three vertices (the corner points) of a triangle are points on a circle, and if one of the three sides of the triangle is a diameter of the circle, then the triangle is a right triangle. In other words, a triangle inscribed in a semicircle must be a right triangle.



5. If two angles and the side between them in one triangle are the same measure as the two corresponding angles and the corresponding side of another triangle, then the two triangles are identical to each other. This is the "ASA rule" of congruent triangles.



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The original proofs that Thales gave for these first five mathematical theorems have been lost because he did not write any books on mathematics and because mathematicians in later years developed more elegant proofs of these results. Nevertheless, Thales' teaching that mathematical theorems must be proven redefined the nature of mathematics significantly. What had been a collection of techniques for measuring and rules for calculating was transformed into a powerful system of rational analysis. His emphasis on the use of logical reasoning from fundamental principles became essential to the study and practice of geometry and remains a basic characteristic of all branches of mathematics today.

Discoveries in Astronomy

In addition to being a philosopher and a mathematician, Thales was an accomplished astronomer. In 585 B.C.E., he correctly predicted an eclipse of the Sun. By studying records that Babylonian

astronomers had kept for many years, Thales was able to determine when the Moon would pass in front of the Sun, blocking it from view in his part of the world. His ability to foretell this event and explain why it occurred amazed his Greek countrymen, who believed that the Sun's disappearance meant that the gods were mad at them. During his lifetime, he was better known for making this accurate prediction than for any of his other accomplishments.

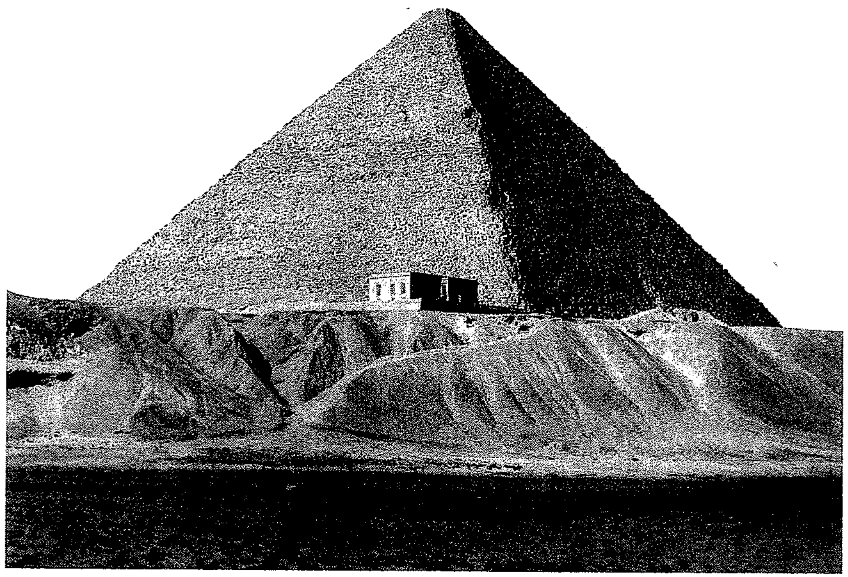
Thales proposed theories to predict and explain the summer solstice—the longest day of the year—and the vernal and autumnal equinoxes—the days in the spring and fall when sunrise and sunset are 12 hours apart. Some historians claim that he wrote an astronomy book about eclipses, solstices, and equinoxes, but no copies of such a book have ever been found.

In addition to studying the Sun, Thales made observations of the stars. The Greeks had identified many groups of stars called constellations that seemed to be arranged as the outlines of the shapes of various animals and people. They gave them names such as Scorpio, Aquarius, Leo, and Gemini; named their months after 12 of these constellations; and developed an intricate theory called astrology that explained how a person's personality and fate were determined by the sign of the zodiac under which he or she was born. Thales did not believe in astrology but was interested in how the positions of the constellations could be used to help sailors determine their location and to guide them to their destinations while at sea. Greek sailors commonly used the constellation Ursa Major, also known as the Great Bear or Big Dipper, as one of their chief navigational guides. Thales identified a new constellation: Ursa Minor, also known as the Little Bear or the Little Dipper. This group of six stars—which included the North Star, one of the brightest stars in the sky—had a more reliable location in the sky. He recommended that sailors rely on this constellation to guide their travels. This recommendation appeared in a book on navigation titled *The Nautical Star Guide*. Although Thales probably developed the theory, scholars believe that the book was written by a contemporary named Phokos of Samos.

Ingenious Solutions to Practical Problems

Thales' reputation as a learned man became widespread. Wherever he traveled, people sought his advice to solve difficult problems. During a visit to Egypt, the Pharaoh asked Thales to determine the height of one of the pyramids. As he thought about how to approach this problem, Thales observed that the shadows of objects in the sun were different lengths at different times of the day. He reasoned that when his own shadow was as long as he was tall, then the pyramid's shadow would be as long as the pyramid was tall. By employing this simple principle, he determined the height of the pyramid successfully.

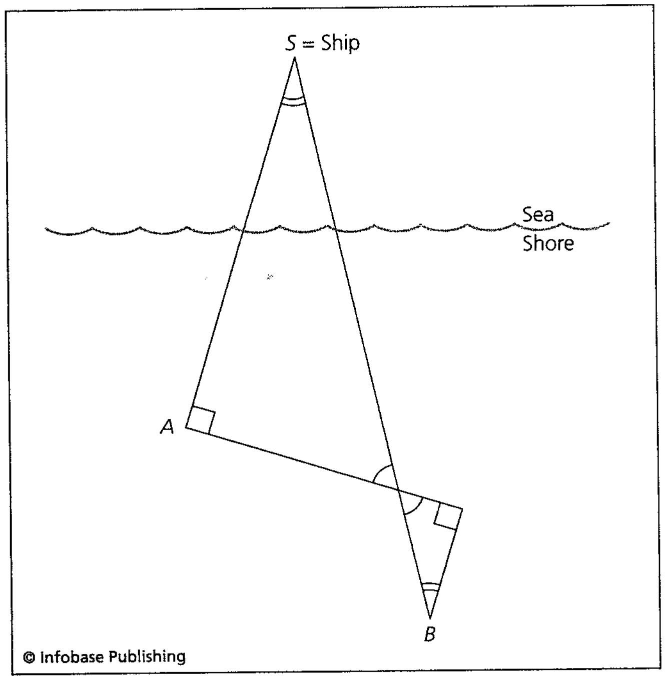
The Greek general Croesus sought Thales' advice to help his army cross the river Halys, which was too wide to build a bridge across and too deep to march through. After some thought, Thales instructed the general to bring all his men and their equipment near the riverbank. He then had them dig a canal behind them in the same direction that the river was flowing. When the ends of the canal were connected to the river, most of the water flowed from the river into the canal behind the army and back into the river further downstream. The army was then able to march through the shallow waters that remained.



Thales determined the height of a pyramid by measuring the length of its shadow. *(Library of Congress, Prints and Photographs Division)*

Merchants and sailors who wanted an accurate way to determine how far a ship was from shore also brought their problem to Thales. Looking from the shore out to a ship that was leaving or entering their port, they could estimate how far away it was based on how small it appeared, but they wanted a more exact way to calculate the actual distance. Thales used his knowledge of similar triangles—triangles of different sizes that each have the same three angles—to develop a method for determining this distance precisely. He knew that for such a pair of triangles, the ratio of two sides of one triangle would be the same as the ratio of the corresponding sides of the other triangle.

The accompanying diagram illustrates Thales' technique. From two points on the shore (labeled A and B in the diagram), observe the location of the ship. Draw a new line through A that forms a



Thales developed a geometrical method to determine the distance from shore to a ship at sea.

right angle with this line of sight, then draw a line through B that forms a right angle with this new line. These lines and the lines of sight from points A and B to the ship will create two similar triangles. By measuring the four sides that are on the shore, one can use the ratios of the corresponding sides to calculate the distances to the ship. The merchants and sailors, who were very good at taking measurements and creating right angles, found this technique easy to use and very valuable.

Legends about Thales

Storytellers preserved and perhaps exaggerated Thales' greatness by creating legends about him, even though some of these stories may not have been true. The philosopher Aristotle told a story that showed how Thales' careful observations of the world allowed him to make a wise business deal. Olives were a very important crop in Greece. In addition to eating olives with most of their meals, Greeks also crushed them to collect olive oil that they used for cooking, as fuel for their lamps, and as an ointment for their skin. Thales had observed that, for several years, the weather had been unfavorable for growing olives. Reasoning that the bad weather could not last much longer, he visited olive orchards, offering to buy the equipment that had been used for crushing olives. The growers, who were in need of money, sold their olive presses to Thales. That year, the weather was excellent for growing olives. When the plentiful crops were harvested and it was time to make olive oil, Thales earned a lot of money renting out the olive presses to the same people who had just sold them to him. Soon afterward, Thales sold the presses back to the growers at fair prices, having demonstrated that he was not just a problem-solver; he could be a successful businessman as well.

Another story that was told about Thales involved a donkey that was used to carry bags of salt from a salt mine. According to this legend, the workers who dug the salt out of the mine would shovel it into sacks that they placed on the backs of donkeys. The animals carried their sacks of salt several miles to the seashore, where workers loaded the bags onto ships. Along the way, the donkeys had to cross a shallow river. One day while crossing the river, one of the donkeys stumbled and fell. As he lay in the river, most of his salt dissolved in the water. When he was able to get up, his load was

much lighter, which made the remainder of the trip much easier for him. After that day, whenever he crossed the river, this donkey would fall in the water, lose some of his salt, and finish the trip with bags that weighed much less than they had originally. The men in charge of the mine asked doctors to examine the donkey to see if it had an injured leg. When no one could determine why the donkey kept stumbling in the river, they finally asked Thales for some help. After observing the donkey for a few days, Thales realized that it was intentionally falling in the water to lighten its load. The next day, Thales filled the donkey's bags with sponges instead of salt. This time, when the donkey fell while crossing the river, the sponges absorbed water, making its bags much heavier. After a few days of carrying wet sponges, the donkey was cured of its bad habit.

The Greek philosopher Plato told a story about Thales and his deep interest in observing the stars. According to this legend, when Thales was out looking up at the stars, he fell into a well. A young girl came by and found him in the well unable to climb out of the deep hole. When Thales told her who he was and what had happened, she laughed at him. She teased the wise man for being so intent on the distant stars above his head that he could not even see what was at his own feet. Plato told this story to make fun of impractical philosophers who were capable of great abstract thoughts but could not do simple things.

Other historians tell another story about Thales and a well that is more credible. In this story, he climbed down into the well to get a better view of the stars. From deep below ground, the walls of the well blocked out the light of the moon and other stars, allowing Thales to better see the stars in the portion of the sky that he wanted to study. In this story, Thales had a sensible reason for climbing down into the well.

Conclusion

Thales died at the age of 78 in approximately 547 B.C.E. During his lifetime, he established the study of natural philosophy, revolutionized the discipline of mathematics, and made contributions to the science of astronomy. His fame as a philosopher, mathematician, astronomer, and ingenious problem-solver was known throughout the entire Greek world. Storytellers made him the central figure in so many stories that his name became synonymous with the word *genius*, much as the name of Albert Einstein is today. His fellow countrymen honored his memory by naming him one of the Seven Wise Men of Ancient Greece—an indication of the respect and admiration that they had for this brilliant problem-solver.

Thales' primary influence on mathematics and science was to establish the need for a theoretical basis and the use of logical reasoning. His natural philosophy introduced the ideas that there are natural explanations for all physical phenomena and that various phenomena are unified by underlying principles. By proving the first theorems in geometry, Thales created a logical structure for the subject and introduced the concept of proof into mathematics. Without these ideas, there would be no modern scientific or mathematical theory; science and mathematics would continue to be a collection of practices that had been observed to work without an understanding of the theoretical principles that explained why things work the way they do.

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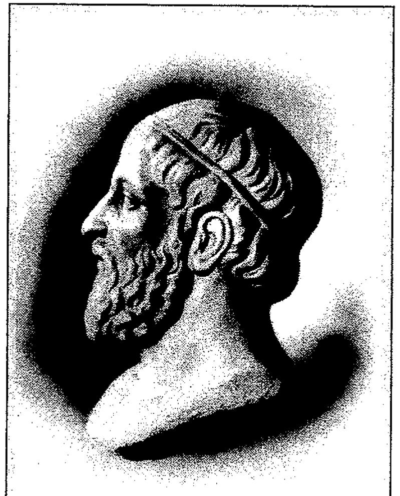
2

Pythagoras of Samos

(ca. 560–ca. 480 B.C.E.)



Pythagoras of Samos made early discoveries in number theory and geometry. *(The Image Works)*

Ancient Greek Proves Theorem about Right Triangles

Pythagoras (pronounced pi-THAG-or-us) of Samos was a mathematician and religious leader in ancient Greece. Conducting some of the earliest work in number theory, he proved fundamental properties about groups of numbers that he termed *perfect*, *friendly*, *odd*, and *triangular*. Pythagoras discovered the mathematical ratios that form the basis of musical theory and proposed that the same ratios exist in astronomy. He gave the first proof of the

Pythagorean theorem about right triangles and, as a result, discovered irrational numbers. His work with the five regular solids illustrated the Greek culture's complicated blending of mysticism and mathematical theory.

First Student is Paid to Learn

Records from historians, mathematicians, and philosophers of the third, fourth, and fifth centuries B.C.E. provide contradictory dates that vary by more than 20 years for Pythagoras's birth, death, and the important events in his life. These sources indicate that Pythagoras was born between 584 and 560 B.C.E. on the island of Samos off the coast of Ionia (present-day Turkey). Although it lay 150 miles east of Athens in the Aegean Sea, Samos was a Greek colony at the time. During the Golden Age of Greece when Pythagoras lived, Samos was a prosperous seaport and cultural center of learning.

Details of Pythagoras's family life are sketchy. Mnesarchus, his father, was a traveling merchant; Pythais, his mother, raised Pythagoras and his two older brothers whose names are not known. At an early age, he showed a talent for arithmetic and music, two interests that he maintained throughout his entire life. Under the guidance of the Greek mathematician Thales, who lived in the nearby city of Miletus, Pythagoras studied mathematics and astronomy. At the age of 20, he traveled to Egypt and Babylonia (present-day Iraq), where he learned mathematics, astronomy, and philosophy—the study of the meaning of life.

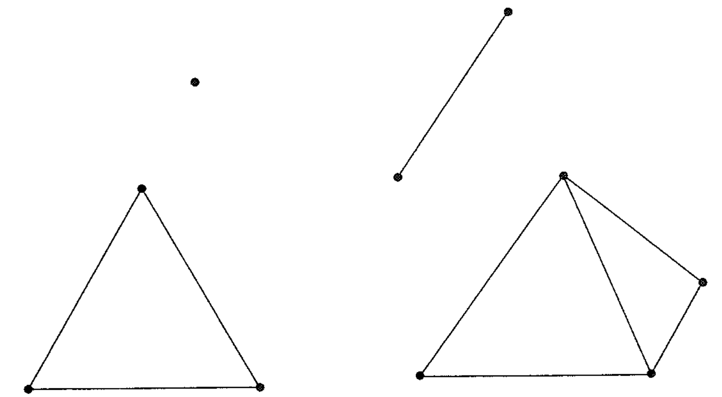
Many legends have been told about Pythagoras, including the story of how he became a teacher. After returning to Samos without any teaching experience and lacking an established reputation as a scholar, Pythagoras was not able to attract any students. Desperate to teach, he offered to pay a young boy to become his first student. Each day he met with the boy on the street, taught him the day's lesson, and paid the boy his daily wage. When this arrangement exhausted Pythagoras's savings and he informed his student that their lessons would have to end, the boy offered to pay Pythagoras to continue teaching him.

Pythagorean Society Blends Mysticism with Mathematics

In 529 B.C.E., Pythagoras moved to Croton, a city in southern Italy, and established a school for adults known as the Pythagorean Society. He divided the men and women of his school into two groups. The *acoustici*, or "listeners," attended his lectures but could not ask questions; they learned solely by listening, observing, and thinking. After five years of studying religion and philosophy, successful listeners joined the advanced group of students. These *mathematici*, or "mathematicians," had obtained enough knowledge to ask questions and express their own opinions. They studied a wider variety of subjects, including astronomy, music, and mathematics. Through Pythagoras's emphasis on arithmetic and geometry, the word *mathematician* eventually came to mean one who studied numbers.

The Pythagoreans, as the members of the society were known, followed strict rules of behavior that reflected their founder's strong convictions. Since Pythagoras believed in reincarnation—the theory that, after people died, they were reborn as different animals—the Pythagoreans ate vegetarian diets, were kind to animals, and never wore wool or leather. They did not eat beans or touch white roosters because they regarded them as symbols of perfection. Valuing generosity and equality, the Pythagoreans shared their possessions and allowed women to participate as both students and teachers. The Pythagorean Society was given credit for all discoveries made by the members, and no written records were kept detailing the activities, teachings, or achievements of the group.

The motto "all is number" expressed Pythagoras's belief that numbers were the fundamental nature of being. He taught that each number had its own distinctive characteristics that determined the qualities and behavior of all things. One was not considered to be a number; it was the essence of all numbers. Two represented women and the differences of opinion. Three represented men and the harmony of agreement. Four, which could be visualized as a square having four equal angles and four equal sides, symbolized equality, justice, and fairness. Five, as the sum of three and two, signified marriage, the union of a man and a woman. As evidenced by expressions



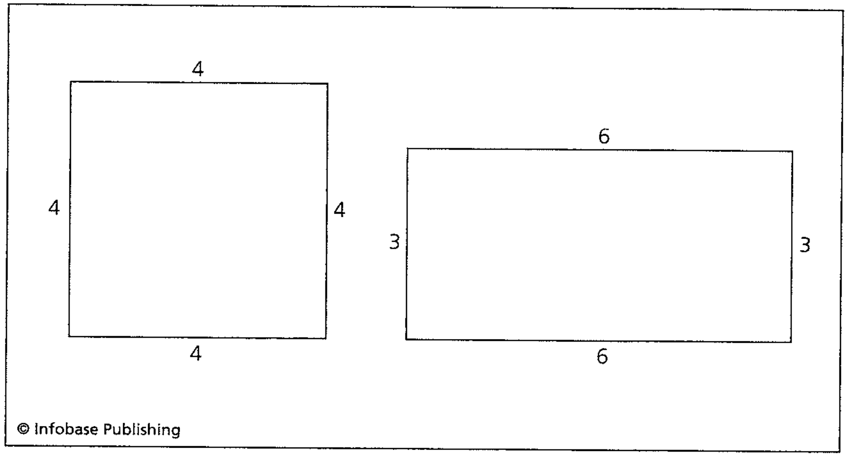
As evidence that "all is number," Pythagoras explained how points determine dimensions.

such as "fair and square" and "a square deal," these Pythagorean ideas became a regular part of Greek language and culture.

Numbers having distinctive mathematical properties fascinated Pythagoras. He called 7 a magical number because it was the only number between 2 and 10 that could not be obtained by multiplying or dividing two of the other numbers. Equations such as $2 \times 5 = 10$; $3 \times 3 = 9$; $8 \div 4 = 2$; and $6 \div 2 = 3$ produced all the numbers in this range except 7. He discovered that 16 was the only number that could be both the area and the perimeter of the same square, a square with all sides of length 4, and that 18 was the only number that could be the area and the perimeter of the same rectangle, a 3×6 rectangle. Pythagoras considered 10 to be holy because it was the sum of 1, 2, 3, and 4, the numbers that defined all the dimensions in the physical world: 1 point represented zero dimensions, 2 points determined a one-dimensional line, 3 points specified a two-dimensional triangle, and 4 points defined a three-dimensional pyramid.

Early Research in Number Theory

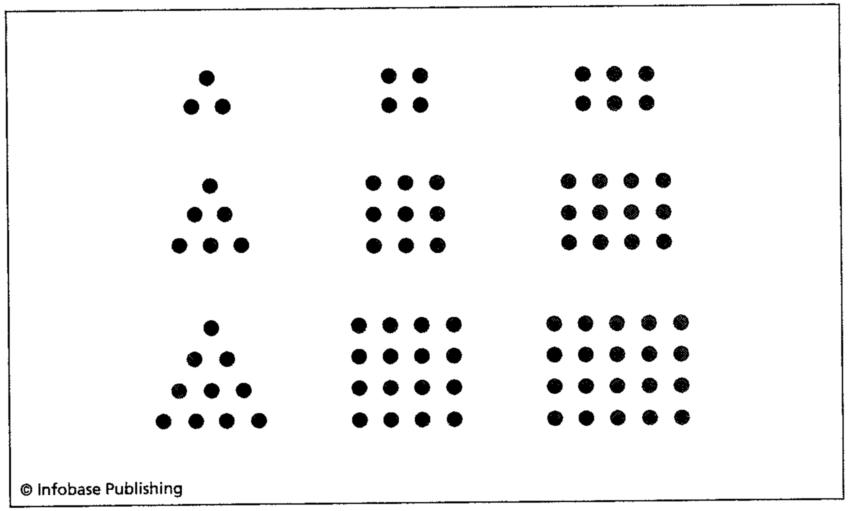
Pythagoras's investigations of numbers extended beyond numerology—the mixture of arithmetic, mysticism, and magic—to the



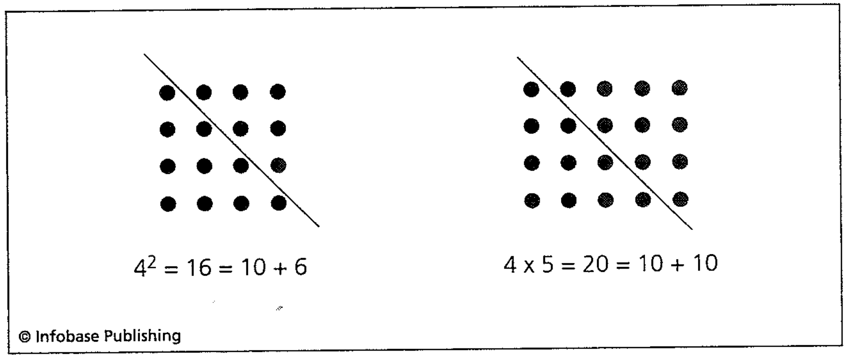
Pythagoras proved that there is only one square and one rectangle whose areas are equal to their perimeters.

branch of mathematics known as number theory. He identified many different groups of numbers based on the arithmetic properties they possessed, such as the concepts of even and odd numbers. A number was even if it could be separated into two equal parts; otherwise it was odd. He further subdivided the even numbers into the even-odds that could be written as two times an odd number (such as $6 = 2 \times 3$), the odd-evens that could be written as more than one factor of two times an odd number (such as $12 = 2 \times 2 \times 3$), and the even-evens that only had factors of two (such as $8 = 2 \times 2 \times 2$).

Pythagoras categorized numbers that could be arranged into similar geometric shapes. He called 3, 6, and 10 triangular numbers and 4, 9, and 16 square numbers because these quantities of dots could be configured into triangular and square patterns. Oblong numbers such as 6, 12, and 20 could be arranged as rectangles with one side being one unit longer than the other. He also studied numbers that could be organized into pentagons (five-sided figures), hexagons (six-sided figures), and other patterns. In addition to identifying various classes of numbers, Pythagoras and his students studied properties of these classes of numbers. They proved that every square number could be written as the sum of two triangular numbers, that every oblong number was twice a triangular number, and many other relationships.

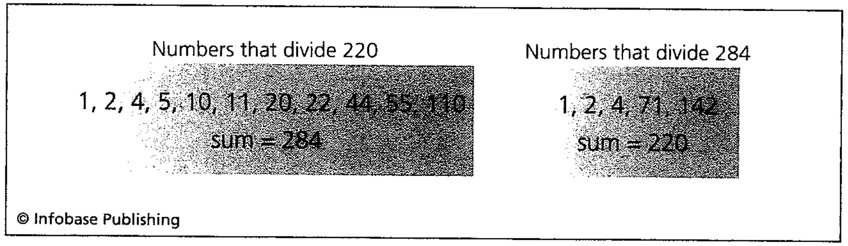


Triangular, square, and oblong numbers take their names from the geometrical shapes that they form.



Pythagoras showed that square and oblong numbers are sums of triangular numbers.

How a number compared to the sum of its factors—those smaller numbers that divided it—determined three more categories of numbers that Pythagoras called perfect, over-perfect, and deficient. A number such as 6 that was equal to the sum of its factors was perfect; 6 could only be divided by 1, 2, or 3 and was equal to $1 + 2 + 3$. An over-perfect (or abundant) number such as 12 was divisible by too many numbers; its factors 1, 2, 3, 4, and 6 added up to more than 12. A deficient number such as 15 did not have enough divisors; its



The divisors of 220 add up to 284, and the divisors of 284 add up to 220.

factors 1, 3, and 5 added up to less than 15. Through his investigations, Pythagoras discovered only four perfect numbers: 6, 28, 496, and 8128. Friendly (or amicable) numbers—two numbers each equal to the sum of the other's factors—were even rarer. The numbers 220 and 284 were the only friendly pair that he was able to identify.

Pythagoras's work with these categories of numbers was the first systematic research in number theory. Modern number theorists continue to study the classes of numbers that he identified. Their work has important applications, including decoding messages and sending files securely over the Internet.

Ratios in Music and Astronomy

In addition to studying whole numbers, Pythagoras studied fractions. He believed that any measurement could be expressed as a whole number or as a fraction (also called a ratio) of two whole numbers. This idea of commensurability formed a basic assumption for his theory that "all is number."

Pythagoras discovered that ratios of whole numbers formed the foundations of musical harmony. As he studied the construction of musical instruments such as the lyre, a stringed instrument like a harp, he noticed that the most pleasing harmonies were produced by plucking strings whose lengths were in simple ratios. A string that was half as long as another produced the same tone but one octave higher. Strings that were $\frac{2}{3}$ and $\frac{3}{4}$ as long as each other produced pleasant-sounding chords called fifths and fourths. He identified the ratios that determined all the notes of the A-B-C-D-E-F-G musical scale.

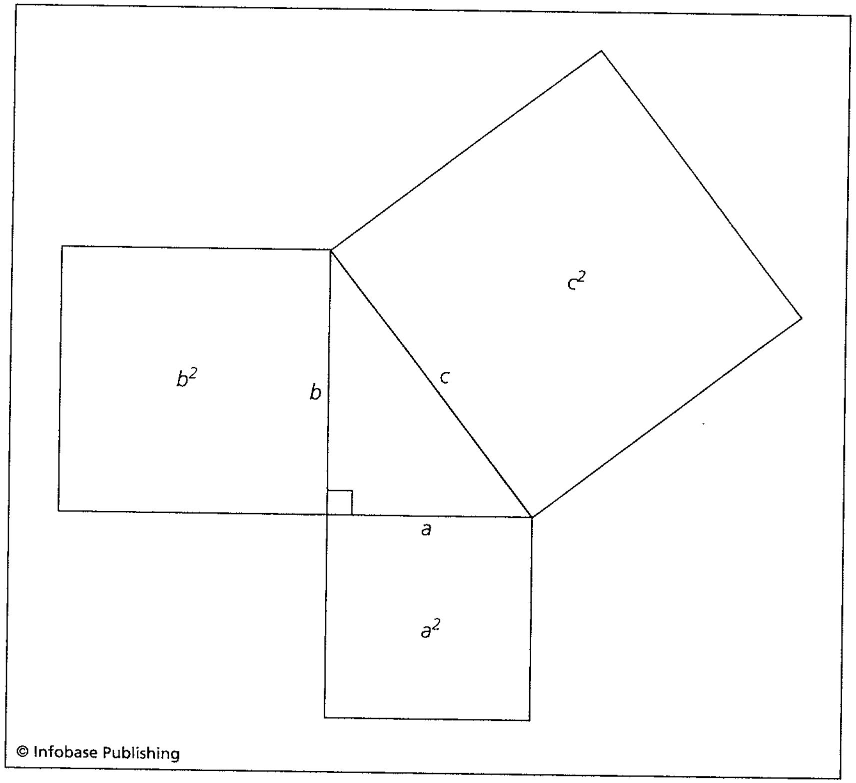
From his observations of the motions of the planets, the Sun, the Moon, and the stars, Pythagoras developed an innovative astronomical theory based on this same pattern of ratios. According to his theory, the universe was a sphere with the stars moving on its outer shell and the Earth sitting at its center. The planets, Sun, and Moon rotated in circular orbits around the Earth. Pythagoras recorded how long it took each body to complete its orbit and determined the radius of each orbit. According to his computations, the distances from the Earth to each of the seven heavenly bodies—the Moon, Mercury, Venus, the Sun, Mars, Jupiter, and Saturn—generated the same ratios as the seven musical notes A through G. He concluded that as the planets moved through the universe they created a natural musical harmony that he called the "Harmony of the Spheres" or the "Music of the Spheres."

Pythagoras's discovery of the mathematical ratios of musical notes remains a fundamental result in the theory of musical acoustics. Although his theory of the Harmony of the Spheres gained widespread acceptance throughout the Greek world, scientists later disproved it. Some of his other work in astronomy, however, was accurate. By observing the curved shadow that the Earth cast on the Moon during a lunar eclipse, he determined that the Earth was a sphere. He also theorized correctly that the Earth rotated on its axis and that the Morning Star and the Evening Star were the same heavenly body.

Pythagorean Theorem

During his travels to Egypt and Babylonia, Pythagoras had learned the well-known property of triangles that if the lengths of the sides of a triangle were 3, 4, and 5, then the triangle had to be a right triangle. The lengths 3, 4, and 5 were related by the equation $3^2 + 4^2 = 5^2$, or $9 + 16 = 25$. The Egyptians were familiar with the principle that, if the sides of a triangle were of lengths a, b, and c and they satisfied the equation $a^2 + b^2 = c^2$, then the triangle had to be a right triangle. They also recognized that every right triangle had to satisfy this equation. Although they had not logically proven these mathematical truths, they accepted them and used them to design buildings, lay out farmland, and plan roadways based on right angles. The Babylonians had also discovered that, for any odd number n, the lengths n, $\frac{(n^2-1)}{2}$ and $\frac{(n^2+1)}{2}$ would form the three sides of a right triangle. Using these formulas, they were able to create an unlimited number of right triangles whose sides were whole numbers such as 3-4-5, 5-12-13, and 7-24-25 triangles.

Pythagoras created the first proof that in every right triangle, the lengths of the sides are related by the equation $a^2 + b^2 = c^2$. This property of right triangles has come to be known as the Pythagorean theorem, and a set of three numbers that satisfies this equation, such as 3-4-5, 8-15-17, or 20-21-29, are called a Pythagorean triple. The Pythagorean theorem is one of the most important results in mathematics. It is used in algebra, where it is the basis for calculating the distance between two points; in analytic geometry, where it



The Pythagorean theorem states that the sides of every right triangle are related by the equation $a^2 + b^2 = c^2$.

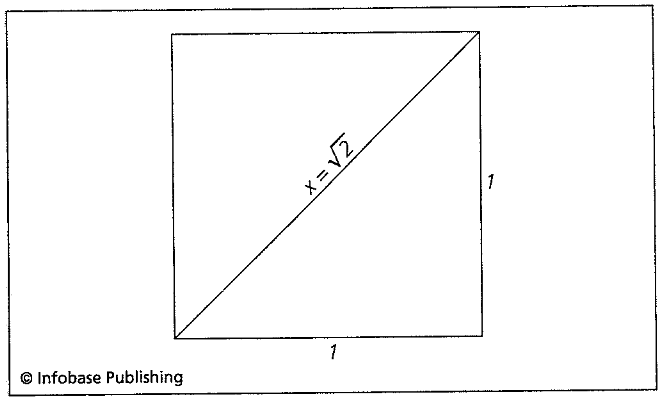
provides the equations of circles, ellipses, and parabolas; in trigonometry, where it describes a fundamental property of sines and cosines; and in many other branches of mathematics. The diagram that Pythagoras used in his proof of the theorem—a right triangle with a square attached to each of the three sides—remains one of the most recognizable images in mathematics.

Irrational Numbers

Pythagoras's work on this theorem led him to the controversial discovery that some numerical quantities could not be expressed as whole numbers or their ratios. He noticed that the diagonal of a square, the line joining two opposite corners, cut the square into two right triangles. If the sides of the square were each one unit long and the diagonal was x units long, then the sides of each right triangle satisfied the equation $1^2 + 1^2 = x^2$, or, more simply, $2 = x^2$.

To estimate the value of this diagonal length x, Pythagoras devised a method of calculating the ratios of the pairs of numbers listed in the following chart. The chart's first row contained the numbers 1 and 1. In each subsequent row, the first number was found by adding the two numbers in the previous row; the second number was found by adding the first number in that row to the first number in the previous row.

Pythagoras discovered that the ratios of these pairs of numbers—$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}$, etc.—provided better estimates for the length of the



The sides and the diagonal of a square are related by the Pythagorean theorem.

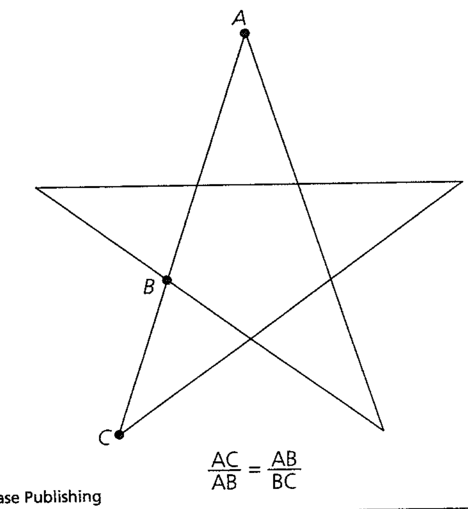
A	B	Ratio of B/A	Decimal value of B/A
1	1	1/1	1.00000
2	3	3/2	1.50000
5	7	7/5	1.40000
12	17	17/12	1.41667
29	41	41/29	1.41379
70	99	99/70	1.41429
169	239	239/169	1.41420

Pythagoras used this chart of integer ratios to estimate the value of $\sqrt{2}$.

diagonal but that this sequence of fractions continued forever without becoming equal to the diagonal length. He ultimately proved that this length, $x = \sqrt{2}$, called the square root of 2, could not be written as a fraction. Working with more and more triangles, Pythagoras and his students found many other lengths, such as $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{6}$, that could not be written as fractions.

The discovery of these irrational numbers (or incommensurables) contradicted Pythagoras's belief that everything in the universe could be expressed in terms of whole numbers and fractions. At first he required that the members of the Pythagorean Society swear an oath not to reveal this discovery to anyone outside the school. According to one legend, when a student named Hippasus broke this code of silence, he drowned mysteriously at sea. In time, Pythagoras reluctantly accepted the existence of irrationals and eventually incorporated them into his further research.

Irrational numbers were a key feature in the five-pointed star, or pentagram, that became the symbol of the Pythagorean Society. Members sewed this geometric design onto their clothing or drew it on the palms of their hands so they could recognize each other. Each point of intersection of two sides of the pentagram cut the sides into lengths that formed a ratio known as the golden mean or golden section. In the diagram below, point B cuts segment AC into lengths that satisfy the equation $\frac{AC}{AB} = \frac{AB}{BC}$. Both fractions in this

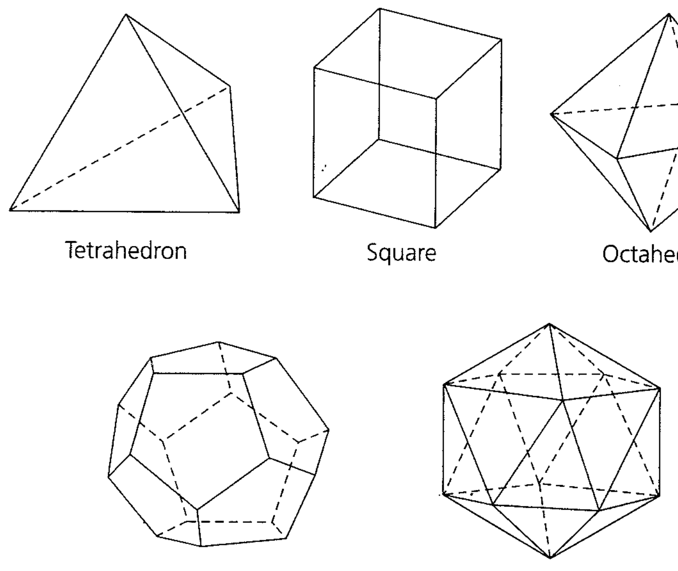


The Pythagoreans adopted the five-sided star known as the pentagram as their official symbol.

equation are equal to the golden mean, a value of $\frac{1+\sqrt{5}}{2}$, or approximately 1.618. Pythagoras and generations of Greek architects and sculptors considered this ratio to be the most beautiful of all proportions. They used it in their designs of many sculptures and buildings; most prominently, in the Parthenon in Athens.

Five Regular Solids

Pythagoras's work with regular solids was another important advance in geometry. A regular polygon, such as an equilateral triangle, a square, or a five-sided pentagon, is a two-dimensional figure in which all the edges have the same length. A regular solid (or polyhedron) is a three-dimensional object in which every side (or face) is the same regular polygon. At the time of Pythagoras, mathematicians knew only three regular solids—a triangular pyramid (or tetrahedron) that combined four equilateral triangles of the same size, a cube that could be made from six equal-sized squares, and a dodecahedron that was constructed



Pythagoras proved that there are only five regular solids.

from 12 regular pentagons. Pythagoras discovered how to make two additional regular solids. He showed that eight equilateral triangles could be combined to create a shape that he called an octahedron and that 20 equilateral triangles could form an object that he called an icosahedron. In addition to discovering these two new polyhedra, he proved that there were no other regular solids. His sophisticated argument, based on a thorough understanding of the geometry of two- and three-dimensional objects, demonstrated how advanced he had become in both mathematical knowledge and logical reasoning.

Although Pythagoras developed the complete theory of regular polyhedra, these five regular solids were eventually named after the great Greek philosopher Plato, who wrote about them 150 years later in his book *Timaeus*. Plato taught that the tetrahedron, the cube, the octahedron, and the icosahedron were the shapes of the atoms of the four elements from which the world was made—fire, earth, air, and water—and that the dodecahedron was the shape

of the universe. In the 900 years that people studied at Plato's Academy, his teachings about these objects were so influential that the five regular solids became known as the Platonic solids.

Around 500 B.C.E., an angry mob burned down the Pythagoreans' school. According to one legend, Pythagoras died in the fire. According to another story, he escaped the fire but was chased by a mob to the edge of a bean field. Not wanting to step on the sacred bean plants, he stopped and was killed by the angry crowd. Other historians reported that he escaped the fire and lived the last years of his life in the nearby city of Metapontum, where he died in 480 B.C.E. After Pythagoras died, his followers started new schools in several other cities, where they carried on his traditions for two centuries.

Conclusion

After their leader's death, the members of the Pythagorean Society made many additional discoveries in mathematics. In algebra, they developed methods for solving more than one equation at the same time. They continued Pythagoras's work in number theory, discovering many properties of prime numbers. The Pythagoreans developed a theory of proportions that expanded the concept of the golden mean. In geometry, they determined how to calculate the sum of the angles in any polygon as well as the sum of the angles outside the polygon. They developed the method of application of areas to construct a square having the same area as a given triangle and introduced the words *parabola*, *hyperbola*, and *ellipse* in the process.

Twenty-four centuries after Pythagoras died, the Mathematical Association of America, one of the professional organizations of college math professors in the United States, chose the icosahedron as its official symbol. This figure that he discovered appears at the top of their stationery and on the covers of all their mathematical journals. Researchers in number theory continue to investigate many concepts that Pythagoras pioneered, including odd and even numbers; triangular, square, and oblong numbers; perfect, over-perfect, and deficient numbers; friendly numbers; and prime numbers. The Pythagorean theorem, irrational numbers, and Platonic solids are tools that modern mathematicians and scientists continue to use in their research.

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3

Euclid of Alexandria

(ca. 325–ca. 270 B.C.E.)

Euclid of Alexandria formulated the principles and techniques that characterized the study of geometry for 2,000 years. *(The Granger Collection)*

Geometer Who Organized Mathematics

The ideas developed by Euclid (pronounced YEW-klid) of Alexandria defined the study of geometry for 2,000 years. *Elements*, his book on geometry and number theory, became the model for the logical development of mathematical theories from first principles and remains the most popular math book ever written. Euclid proved that there are infinitely many prime numbers and developed the Euclidean algorithm for finding the greatest common divisor of two numbers. Attempts to prove his parallel postulate led to the controversial discovery of non-Euclidean geometries in the 19th century. His writings dominated the study of geometry for so many centuries that people simply referred to him as "The Geometer."

Professor of Mathematics

Although he was born of Greek parents, lived in the Greek world, and wrote and taught in the Greek language, the details of Euclid's life are best known from the writings of Arab scholars who lived hundreds of years later. According to these sources, Euclid was born around 325 B.C.E. in Tyre, a large city at the eastern end of the Mediterranean Sea in present-day Lebanon. His father's name was Naucrates, and his grandfather's name was Zenarchus. After living for a number of years in the city of Damascus in present-day Syria, he moved to Athens, the capital of Greece.

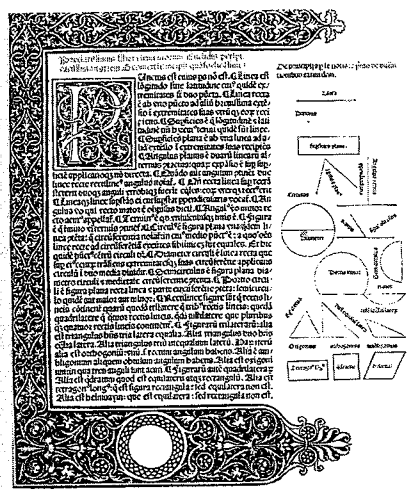
Euclid became a student at the distinguished school that had been established in 387 B.C.E. by the Greek philosopher Plato. Since it was located in the town named Academy just outside Athens, this small but excellent university became known as Plato's Academy. For 900 years, people came from all parts of Greece and from many other countries to learn science, mathematics, and philosophy—the study of the meaning of life—in the tradition of this famous teacher. Plato placed such a high value on the study of mathematics that, according to one legend, he hung a sign over the front door of the academy that read "Let no one ignorant of mathematics enter here." All students at the academy learned advanced mathematics, and most of the accomplished mathematicians of the era studied at this school.

Around 300 B.C.E., Euclid moved to Alexandria, Egypt, where he spent the rest of his life. Because he earned his reputation for the work that he did while living there, he became known as Euclid of Alexandria. This large city at the mouth of the Nile River was the intellectual and commercial center of the Mediterranean world. The warrior-king Alexander the Great established Alexandria in 332 B.C.E. after he conquered the kingdom of Egypt. He and his successor Ptolemy built a massive library where they hoped to collect every book in existence. Whenever learned people came to Alexandria, their books were brought to the library, where scribes made handwritten copies of them. In this manner, the library amassed an extensive collection of more than half-a-million books.

In this city of culture and diversity, Ptolemy also established an institution for research and scholarly activities known as the Museum of Alexandria. This university, which was much larger than Plato's Academy, attracted the greatest minds from every country to discuss, learn, teach, and discover new ideas. Euclid became the first mathematics professor at the museum, where he earned a reputation as a kind and patient teacher. At the museum, he assembled a large group of mathematicians who built a strong reputation for doing research and for discovering new mathematical ideas. Generations of scholars continued this tradition, making the Museum of Alexandria a vibrant research community for 600 years.

Elements

Euclid's greatest achievement was *Elements*, a work in which he organized and presented all the elementary mathematics known at the time. Although Euclid called each of the 13 volumes a "book," they were more like the chapters of a single book. He wrote six chapters on plane geometry, four chapters dealing with properties of numbers, and three chapters on solid geometry. Each chapter included a sequence of propositions and problems. The 465 propositions presented the rules of mathematics stating what conclusions could be drawn from given sets of assumptions. Each proposition was followed by a logical argument called a proof that explained why the proposition was true. Worked-out examples called problems illustrated how to use the propositions in particular situations.

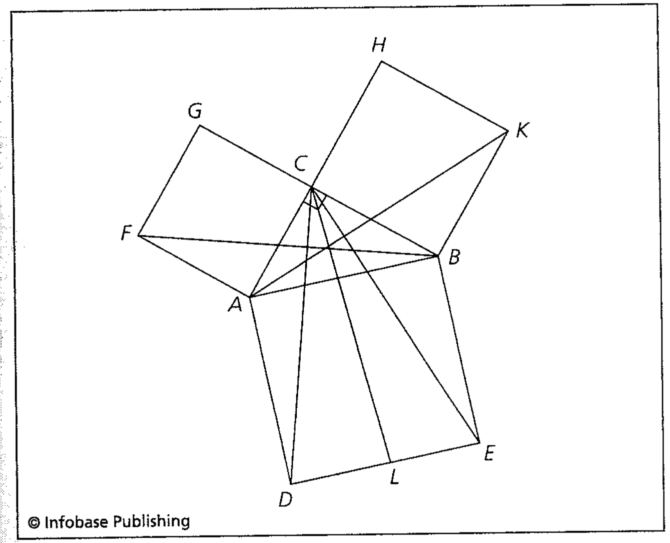


In his famous book *Elements,* Euclid logically developed the theorems of geometry and number theory from basic terms, postulates, and axioms. *(Library of Congress, Prints and Photographs Division)*

In this book, Euclid started from a simple foundation—23 basic terms, five postulates, and five axioms—and developed systematically all the known elementary mathematics of his day. The basic terms were fundamental ideas such as *point*, *line*, and *circle*. Postulates were basic concepts for geometry, such as the idea that through any two points there could be only one line. Axioms, or common notions, were ideas that were fundamental to all mathematics, such as: "Things that are equal to the same thing are equal to each other." Where clear and accurate proofs of propositions existed, Euclid included them. Where known arguments could be improved, he substituted better ones. He organized the material into a meaningful sequence so that each chapter was a coherent unit and the 13 chapters comprised a complete collection.

The first six books of *Elements* presented the rules and techniques of plane geometry. Book I included theorems about congruent triangles, constructions using a ruler and compass, and the proof of the Pythagorean theorem about the lengths of the sides of a right triangle. Book II presented geometric versions of the distributive law $a(b + c + d) = ab + ac + ad$ and formulas about squares, such as $(a + b)^2 = a^2 + 2ab + b^2$ and $a^2 - b^2 = (a + b)(a - b)$. Books III and IV covered the geometry of circles, including results about tangent and secant lines as well as the construction of inscribed and circumscribed polygons. The last two books on plane geometry introduced the theory of proportions and used these results to construct triangles and parallelograms whose sides and areas satisfied specified requirements.

The next four books, or chapters—VII, VIII, IX, and X—presented a collection of ideas in number theory. The first of these books discussed ratios, factors, and least-common multiples of whole numbers always representing each number as a line segment. Book VIII presented results about geometric sequences; plane numbers that have two factors, such as $10 = 5 \cdot 2$; and solid numbers that have three factors, such as $42 = 2 \cdot 3 \cdot 7$. The theorems in Book IX provided results about odd, even, perfect, and prime numbers. Book X, the longest in the work, presented 115 propositions on incommensurable or irrational numbers from a geometric point of view.



Euclid used the famous diagram known as the "bride's chair" to prove the Pythagorean theorem in Book I of *Elements*.

The final three books of *Elements* discussed three-dimensional geometry. Book XI presented procedures for constructing a line through a point that is perpendicular to a plane and for reproducing a boxlike figure known as a parallelepiped. Book XII provided techniques for calculating the volumes of pyramids, cones, cylinders, and spheres. The final book discussed properties of the five regular solids.

Original Results in *Elements*

Most of the material in *Elements* was not original. Euclid built on earlier mathematics texts, also called *Elements*, written in the fourth and fifth centuries B.C.E. by Hippocrates of Chios, Leon, and Theudius. He included the theorems Thales proved about angles, triangles, and circles. The material on proportions came from the work of Eudoxus. The first two chapters on plane geometry, many of the results from number theory, and most of the final chapter on the construction of the five regular solids were primarily the work of Pythagoras.

Mathematicians believe that at least two of the important propositions presented in Euclid's *Elements* were his own discoveries. Proposition 1 in Book VII introduced a technique now called the Euclidean algorithm for determining the greatest common divisor of a pair of numbers, the largest number that divides both of them without leaving a remainder. Using this algorithm, one can find the greatest common divisor of 240 and 55 by making the following sequence of calculations:

240 ÷ 55 leaves a remainder of 20.

55 ÷ 20 leaves a remainder of 15.

20 ÷ 15 leaves a remainder of 5.

15 ÷ 5 leaves no remainder.

Therefore, the greatest common divisor of 240 and 55 is 5. This simple process, one of the oldest-known techniques in number theory, is still presented as an important method of solution in modern textbooks on the subject.

Proposition 20 in Book IX gave Euclid's ingenious proof that there were infinitely many prime numbers—whole numbers like 2, 3, 5, and 7 that cannot be divided by any numbers other than themselves and one. In this proof, he reasoned that if there were only finitely many primes, multiplying them all together and adding 1 would produce a number that either was a new prime or could be divided by some new prime. Since both cases contradicted the original assumption that there were finitely many primes, he concluded that there must be infinitely many primes. Euclid's proof by contradiction was a masterpiece of logic, a classic result that is taught in modern college-level courses in mathematical logic.

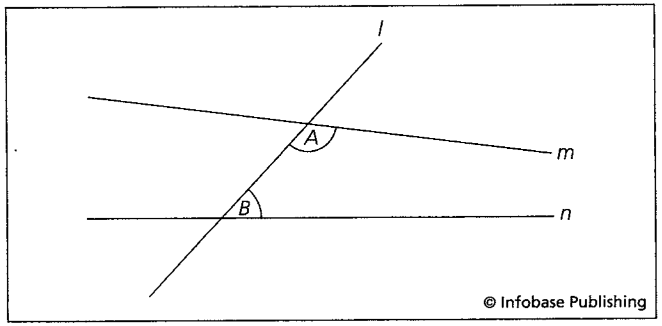
Although other mathematicians had written books similar to *Elements*, none of them had the same impact as Euclid's work. His book set a new standard for mathematical reasoning and explanation. All later mathematical writers embraced his use of logical proofs based on first principles. His ideas on geometry so dominated that branch of mathematics that for centuries mathematicians referred to him as "The Geometer." In the past 2,300 years, more than 1,000 editions of *Elements* have been published in dozens of languages. When the printing press was invented in the 15th century, it was the first math book to be printed. More copies of Euclid's *Elements* have been printed than any other book except the Bible, and more students have used this book than any other textbook on any subject.

Criticisms of Euclid's Methods

Euclid understood that mathematics was useful for solving practical problems such as building bridges, designing efficient machines, and operating successful businesses, but he believed that the real value of mathematics was that it developed a person's mind. Studying mathematics enabled a person to become a disciplined thinker, to make logical arguments, and to appreciate abstract concepts. Mathematics searched for truths that existed outside of the human mind; it was not colored by emotions or opinions. For these reasons, he thought every intelligent person would benefit from a thorough study of mathematics.

His students did not always share Euclid's enthusiasm for the beauty and value of mathematics. According to a popular legend, when a discouraged student asked what he would get from learning mathematics, Euclid told one of his slaves to give the young man a coin so that he could make a profit from his studies. According to another legend, when the emperor Ptolemy attended Euclid's lectures on geometry, he became frustrated by Euclid's thorough and rigorous progression through the material. Accustomed to having clothing, furniture, a chariot, and even royal roads for his exclusive use, Ptolemy asked if there was an easier way to learn the subject. Euclid replied, "There is no royal road to geometry."

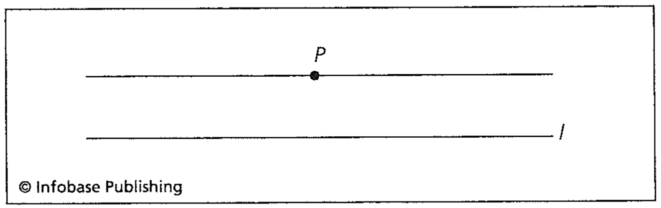
Even other mathematicians criticized Euclid for including in *Elements* the proofs of properties that were clearly true. They argued that it was obvious to anyone, even to a donkey, that the sum of the lengths of any two sides of a triangle had to be greater than the third side. Euclid explained that he proved this principle from other postulates and propositions rather than accepting it as an assumption in order to develop all of elementary mathematics logically from as few basic statements as possible.



Euclid's controversial fifth postulate states that lines *m* and *n* must meet if angles *A* and *B* are less than 180°.

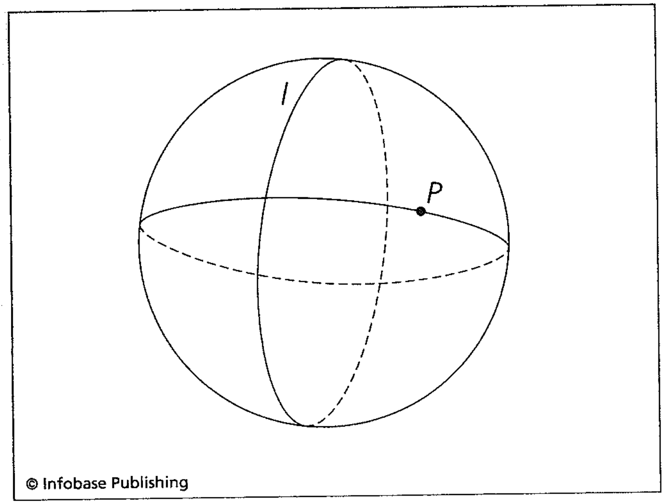
Parallel Postulate

The first four postulates in *Elements* were simple ideas, but the fifth postulate was more complicated. This statement about the angles formed by three intersecting lines meant that, given a point and a line, there was only one line that could be drawn through the point that did not eventually meet the other line. Two such lines that did not meet (or intersect) were said to be parallel. Mathematicians tried unsuccessfully to show that this parallel postulate was really a proposition by attempting to prove that it followed logically from the other postulates. The system of geometry that included Euclid's five postulates became known as Euclidean geometry. In the 19th century, several young mathematicians proved that the parallel postulate was an independent assumption that could not be proven from the other postulates. Substituting different postulates about parallel lines, they created mathematical systems called non-Euclidean geometries.



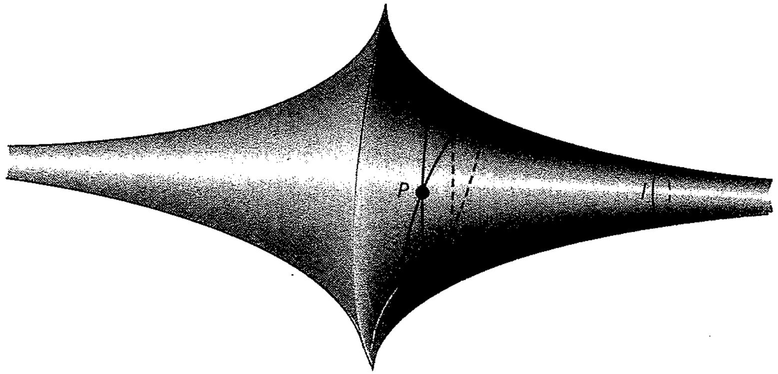
In Euclid's geometry, there is only one line through point *P* that does not meet line *l*.

In 1854, German mathematician Georg Riemann developed a theory about geometry on the surface of a sphere. In this geometry, he defined lines to be the "great circles" that passed through two points on the opposite ends of a sphere. On the surface of a globe, the lines of longitude that pass through the North and South Poles are examples of great circles. In this geometry, there were no parallel lines; any two lines would have to meet in two points. As a consequence, the angles in any triangle added up to more than 180°, rather than exactly 180° as in Euclidean geometry.



In Riemann's geometry, every line through point *P* must meet line *l*.

In 1826, Russian mathematician Nicholas Lobachevsky developed a different non-Euclidean geometry on a pseudosphere—a surface that looked like two trumpet horns glued together. In this geometrical system, there were infinitely many lines passing through a given point that did not intersect a given line. It followed logically that in every triangle, the three angles added up to less than 180°. Hungarian Janos Bolyai in 1823 and German Carl Friedrich Gauss in 1824 also discovered the existence of hyperbolic geometries with infinitely many parallel lines and triangles whose angles summed to less than 180°.

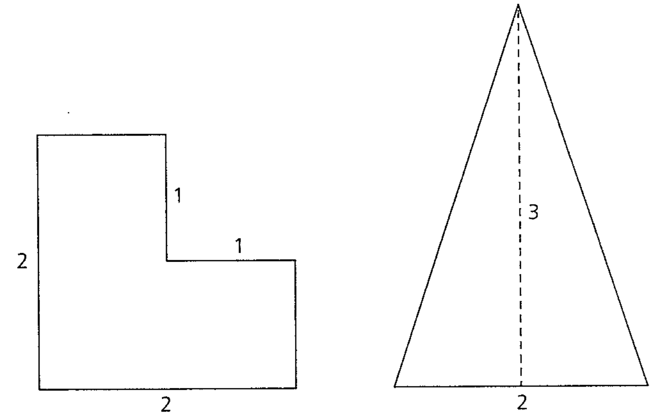


In Lobachevsky's geometry, there are infinitely many lines through point *P* that do not meet line *l*.

Initially, the mathematical community reacted negatively to the discoveries of these non-Euclidean geometries and criticized the mathematicians who discovered them. Eventually, mathematicians realized that non-Euclidean systems were legitimate, that they did not diminish Euclid's work, and that they had practical applications in physics and other sciences.

Euclid's Additional Writings

In addition to writing *Elements*, Euclid also wrote 15 other books on various topics in mathematics and science. His plane geometry text *Data* gave a compilation of facts involving proportions, triangles, circles, parallelograms, and other figures. This work, which may have been a companion book to *Elements*, presented the conclusions that could be drawn from knowing the lengths, areas, or proportions of the various geometrical components in 95 different situations. *On Divisions of Figures* explained how to cut circles, rectangles, and triangles into smaller pieces having particular sizes and shapes. Its 36 propositions showed how to draw a line that cut a triangle into a trapezoid and a triangle having equal areas, how to draw two parallel lines that cut off a desired fraction of a circle, and how to create a rectangle whose



Euclid's *On Divisions of Figures* explained how to create geometrical figures having specified areas and edge lengths. The area and the base of this triangle are equal to the area and the base of a 2 × 2 square from which a smaller 1 × 1 square has been removed.

area was the same as another rectangle from which a square had been removed. The solutions of these types of geometrical puzzles required a deep understanding of the principles of plane geometry.

Two of Euclid's physics books provided the mathematical basis for scientific theories from optics and astronomy. In *Optics*, he discussed the laws of perspective and explained the process of vision. He presented the commonly accepted theory that a person's eye sent out rays that traveled in straight lines toward an object being viewed. Although this theory is incorrect because the human eye receives light rays that are emitted from a light source or are reflected by an object, his mathematical explanations accurately described many other aspects of the process of vision. He explained why objects of different sizes appeared to be the same size when they were viewed at certain angles and why parallel lines appeared to meet. In *Phenomena*, he presented a collection of theorems from spherical geometry and used them to present the geometrical basis for the motion of the stars and planets through the sky. He borrowed much of the material for this book from a similar work called *Sphaerica* (On spheres), written a few years earlier by the Greek mathematician Autolycus of Pitane.

Euclid wrote 11 other books that did not survive and are known only because they were mentioned in the writings of later authors. *Conics* was a four-volume work in which Euclid collected and rearranged all the known theories about the parabolic, elliptical, and hyperbolic curves that arise from slicing a cone-shaped object. Euclid drew most of the material in this work from an earlier work, *Solid Loci*, written by his contemporary Aristaeus. Neither of these texts survived, possibly because by 200 B.C.E. they were superseded by Apollonius's *Conics*, the definitive work on the subject. Euclid's *Surface Loci* was a two-volume work about the geometry of spheres, cones, cylinders, tori, ellipsoids and other surfaces of revolution obtained by rotating a two-dimensional figure about a line known as its axis of revolution. In this work, he discussed curves drawn on such surfaces, as well as the properties of the surfaces themselves. *Elements of Music* discussed the mathematical basis for musical theory, including Pythagoras's ratios for the notes of the musical scale. *Pseudaria* (Book of fallacies) presented a collection of incorrect proofs and common mistakes in logical reasoning from elementary geometry. *Porisms* was a three-volume work containing 38 lemmas and 171 theorems showing how to construct a point, line, or geometrical figure possessing desired properties. Examples of these types of problems included how to find the center of a circle and how to draw a circle that touched three other given circles. Historians discovered additional books on mechanics and music that may have been Euclid's work, but mathematicians who analyzed the style in which they were written believe strongly that other contemporary Greek writers created them.

Conclusion

In the eighth century, when the books written by generations of Greek mathematicians were translated into the Arabic language, Euclid's name was translated as Uclides. When historians discovered these Arabic texts, they noticed that this name was a combination of the Arabic words *ucli*, meaning "key," and *des*, meaning "measure." Some scholars wondered if it was merely a coincidence that the most influential book on geometry, the study of measurement, was written by a man whose name meant "the key to measuring" or if the works of Euclid had been created by a group of mathematicians who published their joint writings under this pen name. Although most mathematicians and historians doubt this theory of group authorship, it has occurred at other times in the history of mathematics. For 300 years after Pythagoras's death, his followers continued to give him credit for all their mathematical discoveries, and in the 20th century, a group of French mathematicians published their joint writings under the name Bourbaki.

Although this theory suggests an interesting possibility, mathematicians are almost certain that there was a real person named Euclid who wrote *Elements*, taught at the Museum of Alexandria, and ultimately died in Alexandria around 270 B.C.E. His masterpiece, *Elements*, defined the teaching of geometry for 2,000 years. Euclid's insistence that all mathematical theorems be proven logically from first principles continues to influence the way mathematicians work today.

At the end of all his proofs, Euclid would write three words meaning "that which was to be proved." In Latin, these words translate as *quod erat demonstrandum*. As a tribute to Euclid, many mathematicians today continue the tradition of ending their proofs with the abbreviation of this Latin phrase—QED.

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4

Archimedes of Syracuse

(ca. 287–212 B.C.E.)

Archimedes of Syracuse used the method of exhaustion to estimate perimeters, areas, and volumes of objects with curved sides. *(Library of Congress, Prints and Photographs Division)*

Innovator of Techniques in Geometry

Archimedes (pronounced ark-i-MEED-eez) of Syracuse established a reputation as an inventor of practical machines but became more famous for his discoveries in mathematics and physics. He developed the method of exhaustion to estimate perimeters, areas, and volumes. Using the Archimedean spiral, he determined tangent lines and trisected angles. Employing experimental techniques, he established theoretical principles about levers, pulleys, and centers of mass. His discovery of the principle of buoyancy established the theory of hydrostatics.

Inventor of Practical Machines

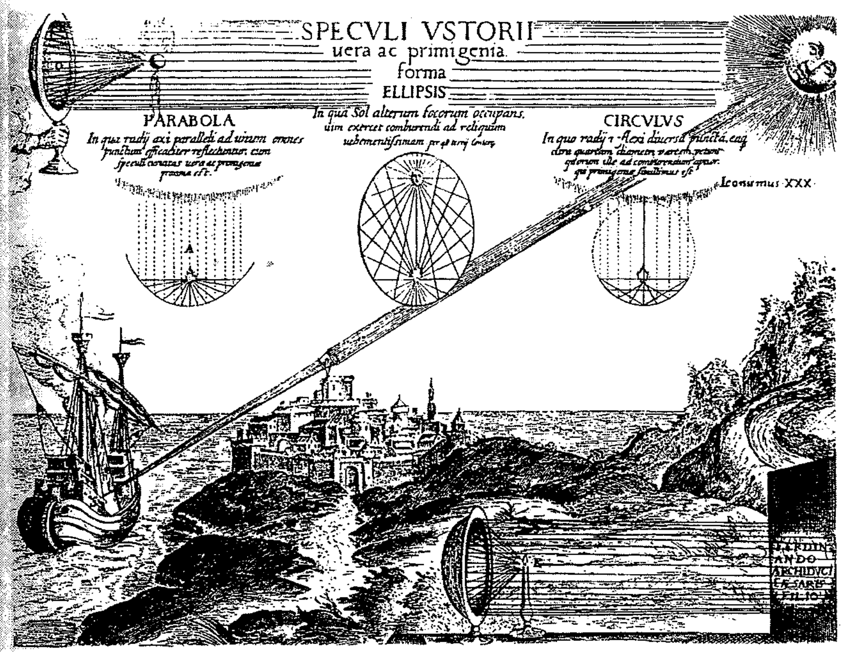
Archimedes was born ca. 287 B.C.E. in the city of Syracuse, an independent Greek city-state on the island of Sicily, off the southwestern coast of Italy. In this cultured city, Archimedes' father, Pheidias, was well known as a respected astronomer. As the son of a scientist and a member of the upper class, Archimedes received a good education. After completing his formal studies in the local schools of Syracuse, he traveled to Alexandria, the great center of learning in Egypt. There he studied under the mathematician-astronomer Canon and the mathematician Eratosthenes, who was head of the Alexandrian library. In this scholarly environment, Archimedes became interested in using mathematics to solve practical problems and in developing new mathematical ideas.

Archimedes quickly established his reputation as a creative inventor. Observing that farmers living near the Nile River did not have an efficient system for drawing water from the river, he designed a large screw enclosed in a long cylinder with a hand crank attached to one end. When the device was placed at an angle with its lower end submerged in water, the spiraling motion of the screw carried water through the device and out the upper end. Egyptian farmers used the water screw, or Archimedean screw, to draw water from the river to irrigate their crops. Various designs of this invention were used throughout the Greek world to drain water from swamps, to empty groundwater from mines, and to pump seawater from the holds of ships.

After living for a number of years in Alexandria, Archimedes returned to Syracuse, where he continued to invent machines and to study the mathematical principles that made them work. Two of the mechanical devices he studied in great detail were levers and pulleys. A lever is a long pole that rests on a pivot point. By pushing down on one end, a person would be able to lift a heavy object on the other end. A seesaw, a crowbar, and the oars of a rowboat are examples of levers. A pulley is a rope wrapped around a wheel. By pulling down on one end, a person is able to lift a heavy object tied to the other end. The rope on a flagpole, a bicycle chain, and a window washer's hoist are examples of pulleys. People had used levers and pulleys for hundreds of years before Archimedes was born, but he was the first person to fully understand the mathematical principles that made these simple machines work. He demonstrated these theories by building intricate machines using combinations of many pulleys and levers that worked in a predictable and exact manner.

Archimedes was so confident of the power of levers and pulleys that he claimed that he could move any object, no matter how heavy it was. He boasted "Give me a place to stand and I will move the earth." His friend King Hieron, the ruler of Syracuse, challenged him to launch a warship loaded with supplies and soldiers. Archimedes rigged up an intricate system of pulleys and levers and, with the slightest effort, set the huge ship into motion to the amazement of the king and the crowd of spectators who witnessed the remarkable feat.

King Hieron asked Archimedes to invent weapons that could be used to defend the walled city of Syracuse from the frequent attacks of the Roman armies. Using the principles of levers and pulleys, Archimedes invented adjustable catapults that could throw 500-pound stones over the walls of the city onto ships entering the



Archimedes designed curved mirrors and lenses that focused the rays of the Sun onto the sails of Roman ships, setting them on fire. *(The Granger Collection)* harbor. He invented huge cranes that could reach over the walls, lift ships out of the water, and drop them back down to sink them. Archimedes devised machines that could shoot many arrows at once. He even invented mirrors and lenses in the shapes of paraboloids, ellipsoids, and hemispheres that could be used to focus the rays of the Sun onto the sails of ships to set them on fire. The Roman soldiers became so terrified of Archimedes' weapons that if they saw a rope hanging over the walls of the city, they would turn and retreat in fear that it might be another of his war machines.

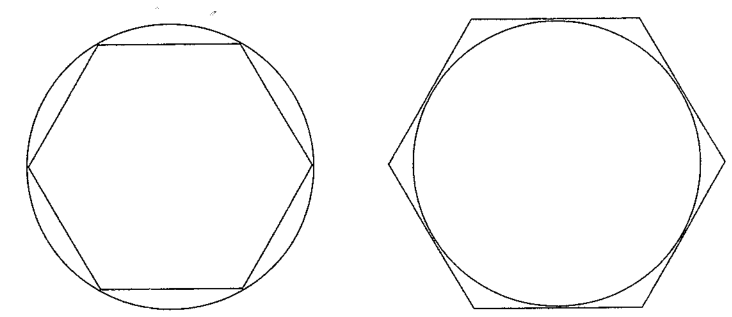
Approximation of Pi Using Inscribed and Circumscribed Polygons

Although Archimedes became famous throughout the Roman Empire for inventing the water screw and many war machines, his mathematical discoveries were much more important. He wrote more than 20 books about his discoveries in diverse branches of mathematics and physics. His book *Measurement of the Circle* introduced new geometrical techniques for calculating distances and areas. *Sand Reckoner* presented innovative strategies to solve arithmetic problems with large numbers. *On Floating Bodies* explained his principle of buoyancy. These books and eight others have been preserved through Arabic and Latin translations. Unfortunately, at least 15 additional books that were mentioned in the writings of other mathematicians and scientists have been lost through the years. Some of his discoveries are known only because he wrote about them in letters that he sent to his friends Canon and Eratosthenes in Egypt.

One of Archimedes' finest mathematical achievements was his perfection of the method of exhaustion. Originally developed in the fifth century B.C.E. by Greek mathematicians Antiphon and Hippocrates of Chios and formalized into a rigorous technique in the fourth century B.C.E. by Eudoxus of Cnidus, this method provided a systematic procedure for estimating areas and perimeters of various shapes using a sequence of simple polygons whose areas or perimeters approximated the shape being measured. Archimedes used the method of exhaustion to estimate the value of the number π (called pi). For centuries, mathematicians had known that the

distance around a circle (its circumference) divided by the distance across the circle (its diameter) was a fixed ratio. Centuries later, this number came to be represented by the Greek letter π, and the relationship was expressed by the formula $\frac{C}{d} = \pi$ or $C = \pi \cdot d$. If a circle's diameter was one unit long (one foot, one yard, one meter), then its circumference would be π units long. Mathematicians knew that this constant π was slightly more than three but had not devised an accurate technique for determining its exact value.

Archimedes used the method of exhaustion to develop a multi-step approach to obtain good approximations for the value of π. He started by drawing a circle of diameter one and locating six equally spaced points on the circle. He connected each point to the next by a straight line to form a six-sided figure inside the circle, called an inscribed hexagon. Since this hexagon was contained inside the circle, the distance around its outer edges (its perimeter) had to be less than the circumference of the circle. Using simple ideas from geometry, he was able to calculate the perimeter of the inscribed hexagon. He knew that this value would be close to, but less than, the value of π. He used the same six points to construct a six-sided figure that was bigger than the circle. By determining the perimeter



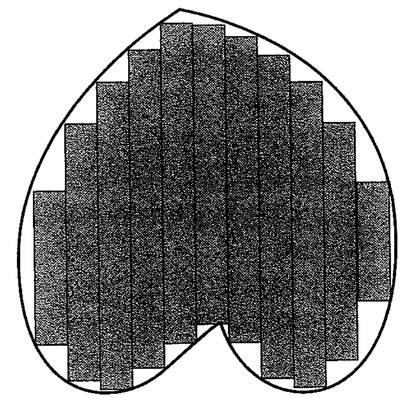
Archimedes estimated the circumference of a circle by finding the perimeters of inscribed and circumscribed polygons.

of this circumscribed hexagon, he found a number that was also close to, but greater than, the value of π. From these two hexagons, he determined that the true value of π was between 3.00 and 3.47.

He repeated this process by constructing 12-sided figures inside and outside of the circle. The perimeters of these two figures revealed that π was between 3.10 and 3.22. Repeating this process with figures having 24 sides, 48 sides, and 96 sides, he determined that π was between $3\frac{10}{71}$ and $3\frac{10}{70}$, that is between 3.1408 and 3.1429. The actual value of π cannot be expressed as a fraction, a mixed number, nor as a number with finitely many decimal places; the digits after the decimal point never end. To four decimal places, its value is 3.1416. Archimedes' approximation was much better than any other estimate known to the Greeks at the time. He published this technique and the numerical results in the book *Measurement of the Circle*. This book was widely circulated, translated into many languages, and used by students of mathematics throughout the Middle Ages. Influenced by Archimedes' writing, mathematicians in the next 18 centuries used this method of inscribed and circumscribed figures with more and more sides to determine the first 35 decimal places of this important number.

Method of Exhaustion to Estimate Areas and Volumes

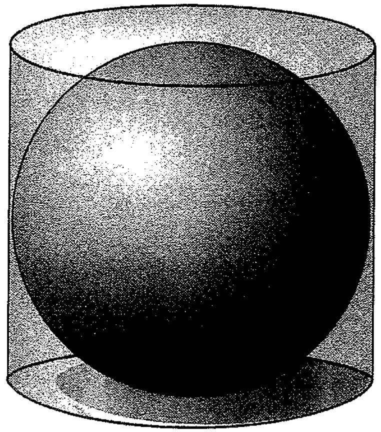
The Greeks who lived at the time of Archimedes knew how to determine the exact area of any geometrical shape having straight sides, such as a hexagon or a trapezoid, by cutting it into a number of rectangles and triangles and adding up their areas. Using the method of exhaustion, they could estimate the areas of figures having curved edges by finding the areas of a sequence of simple geometrical shapes that more closely approximated the shape being measured. Archimedes explained how to use this technique to find the area inside a curve by cutting the figure into slices of equal thickness and fitting into each slice a rectangle that was as large as possible. He used the sum of the areas of these rectangles as a first estimate for the area



Archimedes used the method of exhaustion with a sequence of rectangles to estimate areas of regions with curved boundaries.

inside the curve. Repeating the cutting process with each slice only half as thick as before allowed him to create twice as many thinner rectangles whose areas generated a better estimate for the area inside the curve. By continuing this process as many times as desired, one could estimate the area inside the curve to any degree of accuracy.

In his mathematical writings, Archimedes described three variations of the method of exhaustion using differences, ratios, and approximations of areas and used them to prove a large number of theorems. In his book *On the Quadrature of the Parabola*, he used the method of exhaustion with triangular approximations to determine the area of a segment of a parabola. In another book, *On Conoids and Spheroids*, he showed how to use the method of exhaustion to determine the area inside an ellipse. In the previously mentioned book *Measurement of the Circle*, he used the differences between the areas of inscribed and circumscribed polygons to show that the area of a circle was equal to the area of a triangle whose height was the same as the radius of the circle and whose base was equal to the circumference of the circle. Since the area of a triangle is $\frac{1}{2}$(base)(height), he showed that the area of the circle was $\frac{1}{2}(r)(2\pi r)$, which is the familiar formula $A = \pi r^2$.



Archimedes was most proud of his proof that the volume of a sphere is two-thirds of the volume of the cylinder that contains it.

Archimedes used a modified version of the method of exhaustion to find the surface areas and volumes of three-dimensional objects that had curved surfaces, such as spheres, cones, and cylinders. He presented these results in his book *On the Sphere and the Cylinder*, which he regarded as his favorite among all of his books. He used the method of exhaustion with ratios of triangular areas to find the surface area of a cone. To find the volume of a sphere, he cut the sphere into slices that were the same thickness and fit into each slice the largest possible disk. Since the volume of a disk was easy to calculate, he was able to estimate the volume of the sphere by adding up the volumes of the disks. With thinner and thinner slices, the disks gave better and better estimates, and he eventually determined that a sphere with radius r had a volume of $V = \frac{4}{3}\pi r^3$.

When he discovered this volume formula, Archimedes realized that if the sphere was enclosed in the smallest possible cylinder, like a ball fitting snuggly into a can, the volume of the sphere was two-thirds of the volume of the cylinder. With additional calculations, he discovered that the surface area of the sphere was also two-thirds of the surface area of the cylinder; that is, $A = 4\pi r^2$. Archimedes considered this pair of discoveries to be the greatest achievement of his life. He

was so fascinated by these results that he requested that a picture of a sphere inside a cylinder be engraved on his tombstone along with the fraction $\frac{2}{3}$. Although the location of his grave is no longer known, the Roman historian Cicero wrote that in 75 B.C.E. he had found the location and seen the legendary engraving on the tombstone.

Archimedes was intrigued by the mathematical properties of everyday objects. He studied the surface areas of the curved knives called *arbelos* used by shoemakers and the traditionally shaped bowls called *salinon* that Greeks used to hold table salt. While investigating volumes, he noticed that if a sphere, such as an orange, was cut into slices that had the same thickness, then each slice would have the same amount of orange peel as well. It did not matter if it was cut from the end of the orange or the middle of the orange. If there were 10 slices all having the same thickness, then each piece would have $\frac{1}{10}$th of the orange peel. He presented his analysis of these interesting shapes in his *Book of Lemmas*. In his extensive use of the method of exhaustion to find perimeters, areas, and volumes of curved objects, he developed most of the fundamental concepts of integration, one of the two major ideas in calculus.

Creative Problem-Solver

As a practical scientist, Archimedes found the inspiration for many of his ideas by experimenting with physical models of the objects that he was studying. He would cut out a piece of metal in a particular shape and try to balance it on a stick to find its central axis, spin it on his fingertip to find its center of mass, and weigh it to find its area. With the observations that he made from these informal experiments, he was able to get an idea of what the mathematical solution might be. This experimental approach was radically different from the way that other mathematicians of his time worked. Following the teachings of the great philosopher Plato, they regarded abstract mathematics as the only real path to knowledge. They believed that the physical world and practical experiments would not lead to wisdom and truth. Archimedes was much freer with his thinking and was open to learning how things worked by observing the world around him. In one of his first books, the two-volume work *On the Equilibrium of Planes*, Archimedes presented his discoveries about the laws of levers and the centers of mass of various polygons. As he did in most of his books, he presented only the elegant proofs of his theories without explaining how he had discovered these principles. In the later book *On the Method of Mechanical Theorems*, usually called *The Method*, he described the experimental processes through which he got many of his ideas that he then developed into mathematical theories. This book allowed readers to understand the workings of his brilliant and creative mind.

One of Archimedes' most difficult and important books was *On Spirals*. In this book, he investigated a curve that has come to be known as the Archimedean spiral. This curve starts at a point (called its origin) and expands at a steady rate as it spins around. The formula describing such a curve is $r = a\theta$. Since a spiral could not be created using only a ruler and a compass, the mathematicians who followed Plato's teachings refused to use it to solve problems. Archimedes discovered how to use this spiral to cut any angle into three equal angles—a famous problem called trisecting an angle that mathematicians had been trying to solve for hundreds of years. He was also able to find the equation of the line that was tangent at any point on a spiral curve. A tangent line touches the curve at a specified place and points in the same direction that the curve is headed. Tangent lines incorporate the fundamental concept of the derivative—the second of the two major ideas of modern calculus. With tangent lines and the method of exhaustion, Archimedes almost invented this important area of mathematics 18 centuries before it was eventually discovered by Sir Isaac Newton and Gottfried Leibniz.

Many times during his life, Archimedes demonstrated that he had an incredible ability to see things in a way that no one else did. Part of this ability came from his intense powers of concentration. He was able to block out all distractions and think deeply about a problem for long periods of time. While sitting by the fire on a cold evening, he would often rake some ashes from the fire, spread them on the floor and start drawing diagrams in them to solve a problem that he had been thinking about for days. After taking a bath, Greeks usually rubbed oil all over their bodies. While rubbing in the oil, Archimedes would frequently draw mathematical shapes on

his skin with his fingernails as he continued thinking about an idea that had interested him.

King Hieron asked Archimedes to determine whether his new crown was made from pure gold or if the craftsman had cheated him by substituting some less valuable metal for a portion of the gold. The crown weighed as much as the amount of gold that the craftsman had been given, but no one could think of a way to determine if the crown was pure gold without destroying it. While getting into the tub for his bath, Archimedes noticed that the level of the water rose as he sat down into it, and as more of his body went into the water, the water rose higher and higher. He realized that like any object, his body was replacing an amount of water that was equal to the space it took up (its volume). Archimedes knew he could use this idea to solve the problem about the king's crown. In his excitement over this sudden realization, he jumped from the tub and, without grabbing his towel or putting on his clothes, he ran through the streets shouting "Eureka!" meaning "I have found it!"

When Archimedes arrived at the king's palace, he put the king's crown into a bowl of water and measured how much the water rose. Then he submerged an amount of gold that weighed the same as the crown. When the water did not rise as high as it had with the crown, he was able to determine that the crown was not made of pure gold. One reason that Archimedes was able to make so many discoveries was that his mind was always alert for little hints that gave him the insight to solve big problems.

What Archimedes discovered in the tub was that, when an object is placed into a liquid, the weight of the object will be reduced by the weight of the liquid that it replaces. This principle is now known as the Archimedean Principle of Buoyancy and is a basic law in the science of hydrostatics, the area of physics that deals with properties of liquids. In his book *On Floating Bodies*, he explained the principles of buoyancy and specific gravity and gave a mathematical development of the theory of hydrostatics.

Investigations of Large Numbers

Like his father before him, Archimedes was also interested in astronomy. He constructed a model of the universe with moving parts that showed how the Sun, the Moon, the planets, and the stars moved around the Earth. This planetsphere, which was powered by flowing water, even showed eclipses of the Sun and the Moon. He calculated the distance from the Earth to each of the planets and to the Sun, as well as the size of each heavenly body. He used these measurements to prove a point to other mathematicians who insisted that there was no number large enough to count all the grains of sand on the beach. Archimedes proved them wrong by finding a number that was larger than the number of grains of sand that it would take to fill the entire universe from the Earth to the farthest stars.

In his book *Sand Reckoner*, he explained the process by which he made this huge calculation. He first determined how many grains of sand would equal the size of a poppy seed. Then he estimated how many poppy seeds would equal the size of a finger. Continuing this process, he estimated how many fingers would fill a stadium, how many stadiums would fill a larger space, and so on. Archimedes invented names and notations for these large numbers. By multiplying all these numbers together, he obtained a result that was in the "seventh power of a myriad-myriads." The technique he used to write down such a large number provided the basic idea that other mathematicians used many years later to invent our current exponential and scientific notations. Today we would recognize this huge number as 10^{63}, a one followed by 63 zeros.

Archimedes became famous for his ability to solve complicated problems. His reputation was so widespread that whenever someone had a difficult problem to solve, especially one that involved large numbers, they would call it an Archimedean problem. The name implied that the problem was so hard that it would take someone as brilliant as Archimedes to solve it. One such problem was the Cattle Problem, which involved eight variables representing the number of cows and bulls of four different colors that satisfied eight equations. The solution involves eight numbers that are so large that it would take 600 pages to write them down. Archimedes included the statement of this problem, without its solution, and other similar "recreational" problems in his *Book of Lemmas.*

Less than half of the books Archimedes wrote have survived through the years. He wrote books on various topics in geometry—*On Touching Circles*, *On Parallel Lines*, *On Triangles*, *On the Properties of the Right Triangle*, *On the Division of the Circle into Seven Equal Parts*, and *On Polyhedra*—that are known only because they are mentioned in the writings of other mathematicians. Several of his scientific books—*Elements of Mechanics*, *On Balances*, *On Uprights*, *On Blocks and Cylinders*, and *Catoptrics*—were cited by other scholars but have been lost. A number of works on other topics—*On Data*, *The Naming of Numbers*, and *On Water Clocks*—are also missing.

In 1906, while examining a 12th-century prayer book, a researcher discovered some faint writing in the background that had been partially erased from the parchment. He determined that the underlying text was a 10th-century copy of several of Archimedes' works, including portions of *The Method.* This 174-page book known as Archimedes' Palimpsest is the oldest-existing copy of his written works. In 1998, an anonymous billionaire bought the rare book at an auction for $2 million and loaned it to the Walters Art Museum in Baltimore, Maryland, where researchers continue to clean, preserve, and translate it.

In 212 B.C.E., when Archimedes was 75 years old, the Roman army finally conquered Syracuse. On the day of the Roman invasion, Archimedes was one of the only residents not celebrating at a festival. He was drawing a diagram in the sand to solve a math problem when a soldier ordered him to get up and come with him. Archimedes told the soldier to move out of his light and wait until he finished solving the problem. The impatient and angry soldier killed Archimedes with his spear.

Conclusion

During his career, Archimedes solved almost all of the major problems in mathematics that had been unanswered at the time. With his perfection of the method of exhaustion to estimate areas and his use of the spiral to determine tangent lines, he came very close to inventing calculus 18 centuries before it was ultimately discovered. The experimental approach he used to attack geometry problems challenged the accepted wisdom of his day. His work with curved areas and surfaces advanced tremendously the state of geometry. The calculations he performed with very small and very large numbers introduced new techniques in arithmetic. Because his many original and significant discoveries demonstrated such powerful insight, mathematicians rank Archimedes with Sir Isaac Newton and Carl Friedrich Gauss as one of the three greatest mathematicians of all time.

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Hypatia of Alexandria

(ca. 370–415 C.E.)

Hypatia of Alexandria wrote commentaries on classic mathematical works. *(Library of Congress, Prints and Photographs Division)*

First Woman of Mathematics

The Greek mathematician and philosopher Hypatia (pronounced hi-PAY-shuh) of Alexandria is the first woman known to have taught and written about mathematics. Her commentaries enhanced and preserved classic works of ancient mathematicians. She was a Neo-Platonist philosopher, a teacher of mathematics, and a respected scientist. Her brutal murder by an angry mob marked the end of seven centuries of intellectual culture in Alexandria, Egypt.

The "Perfect" Human Being

Hypatia was born in the city of Alexandria, Egypt, during the last half of the fourth century. The details of her life are known primarily from four sources: *The Letters of Synesius of Cyrene*, which includes several pieces of correspondence from her student Synesius; an excerpt from the historian Socrates Scholasticus' fifth-century work *Ecclesiastical History;* an entry in *The Chronicle of John, Coptic Bishop of Nikiu*, written in the seventh century; and a passage from the 10th-century encyclopedia *Suda Lexicon*. The conflicting information from these historical records places her date of birth between 350 and 370 C.E.

Seven centuries earlier, when the warrior-king Alexander the Great conquered the kingdom of Egypt, he decided to build a great city at the mouth of the Nile River. He designed the city to be a military stronghold, a hub of international commerce, and the world's greatest center of knowledge and learning. He and his successor, Ptolemy I, built a large library with the goal of collecting every book that had ever been written. They established a policy that whenever learned people came to Alexandria, their books would be brought to the library, where scribes would make handwritten copies of them. The copies were then placed in the library, where they were made available to the public. Ptolemy also established the Museum of Alexandria as a university where scholars from every country could gather to discuss, learn, teach, and discover new ideas. Many important discoveries were made by Greek mathematicians living and working in Alexandria.

Hypatia's parents were among the well-educated citizens of Greece who were attracted to this beautiful city at the center of the civilized world. Her father, Theon, was a professor of mathematics and astronomy at the museum. In addition to teaching, he wrote about eclipses of the Sun and the Moon and edited existing mathematics and astronomy textbooks to make the material more accessible to his students. He was eventually appointed to the position of director of the museum. Hypatia was an only child; her mother died when she was very young.

Theon devoted himself to the idealistic goal of raising his daughter to become the "perfect" human being, enabling her to

reach the full potential of her physical, mental, and spiritual abilities. Following a fitness routine that her father had devised for her, Hypatia spent many hours each day running, hiking, horseback riding, rowing, and swimming. Theon often accompanied her in these physical activities. Her father also designed a challenging educational program to develop her mental abilities. Under his instruction, Hypatia learned to read and write, do math and science, debate, and speak in front of an audience. Accompanying her father to the museum each day, she read the classic works of Greek literature and was exposed to the ideas of the ancient philosophers and scholars.

In the environment that the museum, library, and city provided to her, Hypatia became an excellent public speaker and excelled in mathematics and philosophy—the study of the meaning of life. To complement her academic education, she traveled to Greece and to other countries around the Mediterranean Sea. As she visited these countries and met many new people, she developed an understanding of various cultures and a respect for different traditions and points of view.

Commentaries on Classical Mathematics Books

When she returned to Alexandria, Hypatia joined her father at the museum, where she taught courses in mathematics and philosophy. Although she rapidly established her reputation as an excellent teacher and attracted a loyal following of students, her mathematical writings had a greater impact on future generations of students. Hypatia worked with her father to revise and update classic mathematics texts. In her time, these were called "commentaries"; today such works would be called "edited versions." Writers of commentaries made corrections, revised some explanations, and added material that had originally been presented in other books that were no longer available. They updated the books by including new discoveries that had been made since the time that the books had first been written. Hypatia, Theon, and other professors used these new books to teach their students at the museum. Traveling scholars brought copies of the commentaries to universities in other countries, where they were translated into Latin, Arabic, and other languages.

Hypatia and Theon worked together to produce a commentary on Euclid's *Elements.* This book, considered by academic scholars to be the most influential textbook ever written, had been created 700 years earlier by the museum's first mathematics professor, the Greek scholar Euclid. In 13 chapters that Euclid called "books," he had logically and systematically presented all the elementary mathematics that was known at the time so university students could study from a single textbook. Theon and Hypatia corrected mistakes that had been made in earlier copies of the book and expanded some explanations to make the material easier to understand. They also hoped to preserve this mathematical knowledge for future generations. The edition of *Elements* that they prepared was so highly regarded that it became the standard edition of the text for the next thousand years. Although hundreds of versions of Euclid's *Elements* were prepared by other mathematicians throughout the centuries, Theon and Hypatia's edition remained the one most frequently used and was considered to be the most faithful to the original manuscript.

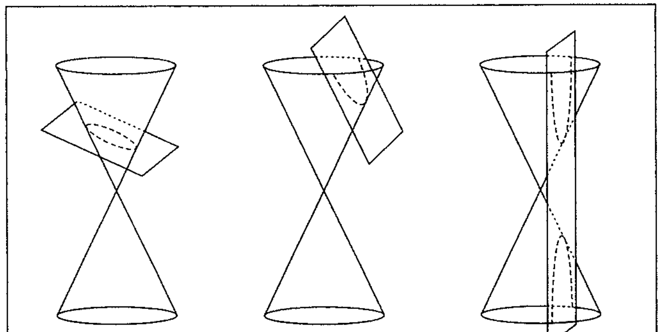
Independent of her father, Hypatia wrote commentaries on three other mathematics books—Diophantus's *Arithmetic,* Ptolemy's *Handy Tables,* and Apollonius's *Conics.* In these important mathematical works from three different centuries, each author had presented the most advanced knowledge in a particular branch of mathematics. Hypatia's broad understanding of mathematics and its applications, as well as her experience as a teacher, enabled her to improve upon the versions of these works that existed in her day.

Hypatia's first commentary was on the book *Arithmetic.* Written by the Greek mathematician Diophantus around the year 250, this book presented a collection of 150 word problems drawn from all areas of mathematics. After stating each problem, the author presented one or more mathematical equations that represented the relationships between the unknown quantities and gave a method of solution using techniques from algebra. In this book, Diophantus introduced a systematic notation for representing exponents beyond the square and cube as well as a method for dealing with coefficients. Hypatia added two types of material to Diophantus's work. She included a technique for solving a pair of simultaneous equations—two equations relating the same variables that needed to be solved at the same time. In modern algebraic notation, the system of equations is represented as $x - y = a$ and $x^2 - y^2 = m(x - y) + b$ for specified values of the constants a, b, and m. Historians do not know whether she invented this method or whether it was discovered by other mathematicians after *Arithmetic* was first written. She also added steps at the end of many problems showing readers how to verify that their solutions were correct.

Hypatia wrote another commentary on the book *The Astronomical Canon* that had been written by the astronomer Ptolemy around the year 150. This book, also published under the title *Handy Tables,* contained lists that gave the lengths of the arcs of a circle that were cut out by angles as small as $\frac{1}{3600}$ of a degree. These calculations were used by astronomers, sailors, land surveyors, and others whose work involved geometry. Theon, who had published an earlier commentary on these tables, stated that the quality of his daughter's work surpassed his own. Historians disagree about whether Theon's praise for Hypatia's work was sincere or whether he was attempting to enhance her reputation as a scholar.

The third commentary written by Hypatia was on the book *Conics* that had been written by the Greek mathematician Apollonius around 200 B.C.E. This book described how the three important curved shapes—the ellipse, the parabola, and the hyperbola—could be obtained by slicing a double cone with a flat plane. An ellipse describes the orbit of a planet around the Sun as well as the path traveled by an electron in an atom. A parabola is the shape used to design reflectors for flashlights and cables for suspension bridges. Hyperbolas

Apollonius's *Conics* was one of three classic works for which Hypatia produced a valuable commentary. *(Library of Congress, Prints and Photographs Division)*



The intersection of a plane and a double cone will produce one of three conic sections—ellipse, parabola, or hyperbola.

are used in the design of cooling towers at power plants; they also describe the paths of some comets. All three curves are used in the design of antennae, telescope lenses, and television satellite dishes. This book remained the most advanced work on these important curves for another 1,300 years.

Famous Teacher, Philosopher, and Scientist

In addition to becoming well respected for her mathematical writings, Hypatia developed a reputation as an excellent speaker and teacher. She gave public lectures and private lessons on mathematics and philosophy. When she taught or spoke, Hypatia would dress in the long, flowing robes traditionally worn by philosophers of her time. She taught her students to have respect for different perspectives and competing points of view on controversial issues. Her teachings on philosophy incorporated the ideas of Plato, who encouraged people to seek knowledge and to develop their spiritual side, as well as the ideas of Aristotle, who emphasized logic and the analysis of the physical world.

After earning a strong reputation as a teacher at the museum, Hypatia became the head of another school in Alexandria—the

Neo-Platonic School of Philosophy. Neo-Platonists believed that the goal of life was to focus less on the physical world of the body and more on the higher spiritual world of the mind and the soul. People traveled to Alexandria from many different countries to hear Hypatia speak and to learn from her. Her home and her school became gathering places where educated people discussed and learned mathematics and philosophy.

In Alexandria at the time of Hypatia, many women were well educated, but few women taught at universities, and it was very rare for a woman to be a leader in her field. Hypatia's accomplishments as a respected leader in two fields—mathematics and philosophy—indicated her stature as a leading intellectual of her era.

Hypatia was also a respected member of her community. She was asked by friends to speak to government officials on behalf of needy people. The citizens of Alexandria knew her to be a generous, kind, and caring individual. She never married, freeing her to devote her life to writing, teaching, and works of charity.

Hypatia was also a capable scientist who possessed practical skills. She designed two scientific instruments for her friend and former student Synesius, who eventually became the Christian bishop of the city of Ptolemais. A collection of the letters that he wrote to friends and colleagues was published as *The Letters of Synesius of Cyrene*. In his letters to Hypatia, Synesius thanked her for designing an astrolabe and a hydrometer for him. An astrolabe was an instrument used by sailors to determine a ship's location by measuring the positions of the stars. Hypatia did not invent the first astrolabe; it had already been in use for a hundred years. As a scientist and teacher, Hypatia was able to clearly explain to her friend the instructions for making and using an astrolabe. A hydrometer was an instrument that measured how heavy a liquid was compared to an equal volume of water. Historians believe that Synesius probably used the hydrometer that Hypatia designed for him to mix his own medicines or to diagnose his health problems.

Brutally Murdered

By the early fifth century, the city of Alexandria became involved in rapid social changes. The Museum of Alexandria, its valuable collection of books, and the intellectual culture that surrounded it were no longer priorities for the Romans who ruled Egypt nor for the residents of Alexandria. Political leaders struggling to maintain their authority felt threatened by the large groups of enthusiastic followers who attended private meetings at Hypatia's home, lectures at her school, and speeches that she gave in public gatherings throughout the city. Local leaders of the Christian and Jewish churches thought that her mathematical and scientific ideas contradicted the teachings of their religions and that her philosophical ideas were attracting the followers of their religions.

In 415, Hypatia became entangled in a dispute between a group of Christians led by Cyril, the archbishop of Alexandria, and the supporters of Orestes, the government prefect of Alexandria. After many members of both groups suffered violent deaths, the feud reached its peak. As Hypatia was riding through the streets of Alexandria in her chariot on her way to giving a speech, an angry mob surrounded her. They dragged her from her chariot, beat her, and threw her to the ground. They tore off her clothes, cut her body into pieces, and burned them. The dispute was

An angry mob dragged Hypatia from her chariot and murdered her. *(ARPL/Topham/The Image Works)*

promptly settled, but no one was ever arrested nor punished for this violent incident.

Conclusion

Hypatia's death marked the end of the era of intellectual enlightenment and the advancement of knowledge that had thrived in Alexandria for 750 years. After she was murdered, many scholars left the city and moved to Athens or to other centers of learning. In the next decades, foreign invaders and rebellious citizens attacked the buildings of the great university and vandalized the library, burning many books to heat the water at the public baths. Theon and Hypatia's version of Euclid's *Elements* and Hypatia's commentaries on Diophantus's *Arithmetic*, Ptolemy's *Handy Tables*, and Apollonius's *Conics* were preserved only through the copies that had been brought by scholars to cities in the Near East, where they were translated into Arabic. Any philosophical writings that she may have created were permanently lost.

Hypatia's uniqueness as an intellectual woman in a male-dominated culture and the violent manner of her death have caused her story to be retold by historians and writers through the centuries. Historical texts from the fifth, seventh, and 10th centuries tell the story of her life and death and of her contributions to mathematics and philosophy. In 1851, English novelist C. Kingsley dramatized the story of Hypatia's life and of her murder in his book *Hypatia*. Brief biographical portraits were included in popular collections of short stories such as E. Hubbard's 1908 book *Little Journeys to the Homes of Great Teachers*. In the 1980s, modern-day scholars established the journal *Hypatia*, in which they publish scholarly papers written by women about issues in philosophy and women's studies.

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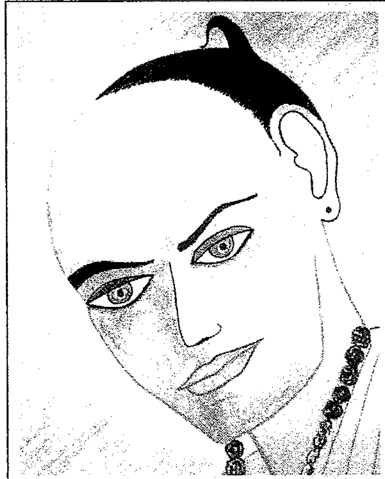
Āryabhata I

(476–550 C.E.)



Āryabhata wrote an influential treatise on mathematics and astronomy. *(Dinodia/The Image Works)*

From Alphabetical Numbers to the Rotation of the Earth

Āryabhata I (pronounced AR-yah-BAH-tah) wrote one of India's most enduring treatises on mathematics and astronomy. The alphabetical system of notation he developed used combinations of consonants and vowels to represent large numbers. He presented efficient methods for calculating cube roots, formulas for summing series of numbers, and algebraic methods for solving indeterminate linear equations. The table of sines and the estimate for π that he promoted remained in use for centuries. In